

Recall: Every permutation can be written as a product of disjoint cycles.

Given any $\sigma \in S_n$, we may write

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ i_1 & i_2 & i_3 & i_4 & \dots & i_n \end{pmatrix}$$

To construct the product of disjoint cycles:

(1) Start at 1 in the first row. Construct a cycle by writing:

$$\sigma_1 = (1 \ i_1 \ i_{(i_1)} \ i_{(i_{i_1})} \ \dots \ i_{(\dots i_1)})$$

(i>1) (j) Choose the smallest $k \ 1 < k \leq n$ which does not appear $\sigma_1, \dots, \sigma_{j-1}$

If such a k exists define $\sigma_k := (k \ i_k \ \dots \ i_{(\dots i_k)})$

If not define $\sigma := \sigma_1 \dots \sigma_{j-1}$.

We can also write any permutation as a product of transpositions, i.e. elements of the form (ij) for some $i, j \in \{1, \dots, n\}$ and $i \neq j$.

Ex: $(24758) = (28)(25)(27)(24)$

$(3527) = (37)(32)(35)$

The parity of a permutation is either even or odd and this is well-defined.

Any way of writing (24758) must have decomposition into an even number of transpositions. We say (24758) has even parity or is an "even permutation".

Any way of writing (3527) must have decomposition into an odd number of transpositions. We say (3527) has odd parity or is an "odd permutation".

Note: The inverse of a transposition is itself. $(ij)^{-1} = (ij)$

Suppose that $\sigma = \tau_1 \dots \tau_{2k} \leftarrow \tau_i, \rho_i$ are transpositions.

$\sigma = \rho_1 \dots \rho_{2m+1}$

$\sigma \sigma^{-1} = \rho_1 \dots \rho_{2m+1} \rho_{2m+1} \dots \rho_1$

$\sigma^{-1} = \rho_{2m+1} \dots \rho_1$

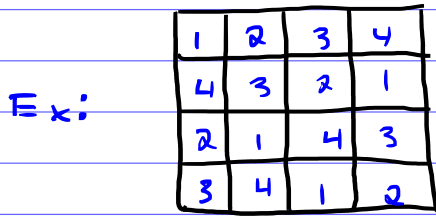
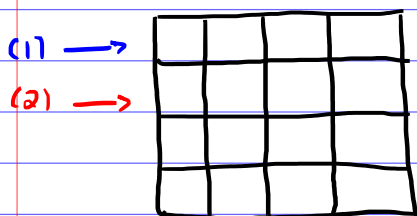
Therefore, identity = $\sigma \sigma^{-1} = \tau_1 \dots \tau_{2k} \rho_{2m+1} \dots \rho_1$. Then the identity is a product of an odd number of transposition.

Transpositions switch two elements.

The identity: In any form, if I switch i and j with a transposition, I have to eventually switch them back.

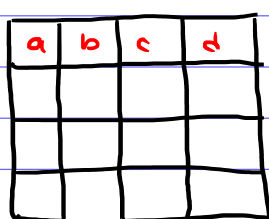
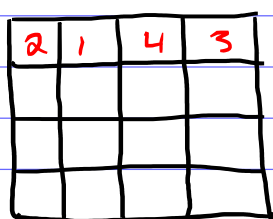
So the identity can only be written in terms of an even # of transpositions.

1(a) 4x4 Latin Squares.



(1) The first row corresponds to all permutations of the set $X = \{1, 2, 3, 4\}$ and there are 24 possibilities.

(2) The second row, we cannot have two numbers that are same in the column. Take any permutation in the first row.

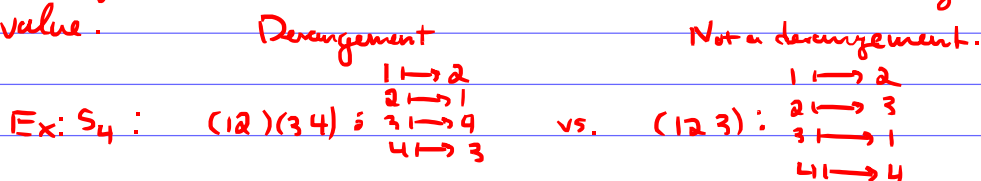


The permutation I can put in the second row cannot fix any letter in the first (Derangement)

$$d(4) = 4! \sum_{i=0}^4 \frac{(-1)^i}{i!} = 4! \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \right) = 24 - 4 + 1 = 9$$

of possibilities in the second row is 9.

Derangement is a permutation which does not fix any value.



(3) Depends on the derangement chosen.

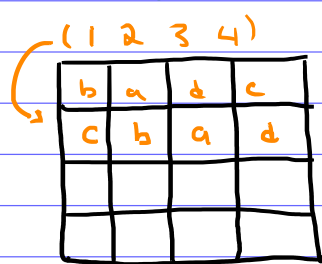
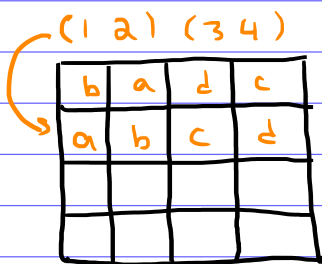
S_4 "double transpositions"
"4-cycles"

Double Transposition.

- (12)(34)
- (13)(24)
- (14)(23)

6 4-cycles

- (1234) (1423)
- (1243) (1432)
- (1324)
- (1342)



Possible 3rd Rows:

- 3 4 1 2
- 4 3 1 2
- 3 4 2 1
- 4 3 2 1

- 3 2 1 4
- 4 3 2 1

$24 \times 3 \times 4$ + $24 \times 6 \times 2$

576 possible Latin squares.