Week 9 Section (5/26/21)

Recall: Every permutation can be written as a product of disjoint cycles. Given any $\sigma \in S_n$, we may write

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ i_1 & i_2 & i_3 & i_4 & \cdots & i_n \end{pmatrix}$$

To construct the product of disjoint cycles:

(1) Start at 1 in the first row. Construct a cycle by writing:

$\sigma_1 = (1 \; i_1 \; i_{i_1} \; i_{i_{i_1}} \; \cdots \; i_{i_{\cdots i_1}})$

$(j>1)$ (j) Choose the smallest k $1 < k \leq n$ which does not appear $\sigma_1, \ldots, \sigma_{j-1}$

If such a k exists define $\sigma_k := (k \; i_k \; \cdots \; i_{i_{\cdots i_k}})$

If not define $\sigma := \sigma_1 \cdots \sigma_{j-1}$.

We can also write any permutation as a product of transpositions, i.e. elements of the form $(i\,j)$ for some $i, j \in \{1, \ldots, n\}$ and $i \neq j$.

Ex: $(2\,4\,7\,5\,8) = (2\,8)(2\,5)(2\,7)(2\,4)$

$(3\,5\,2\,7) = (3\,7)(3\,2)(3\,5)$

The parity of a permutation is either even or odd and this is well-defined.

Any way of writing $(2\,4\,7\,5\,8)$ must have decomposition into an even number of transpositions. We say $(2\,4\,7\,5\,8)$ has even parity or is an "even permutation".

Any way of writing $(3\,5\,2\,7)$ must have decomposition into an odd number of transpositions. We say $(3\,5\,2\,7)$ has odd parity or is an "odd permutation".

Note: The inverse of a transposition is itself. $(i\,j)^{-1} = (i\,j)$

Suppose that $\sigma = \tau_1 \cdots \tau_{2k}$ ← τ_i, ρ_i are transpositions.

$\underline{\sigma = \rho_1 \cdots \rho_{2m+1}}$ $\qquad \sigma\sigma^{-1} = \rho_1 \cdots \underbrace{\rho_{2m+1}\rho_{2m+1}} \cdots \rho_1$

$\sigma^{-1} = \rho_{2m+1} \cdots \rho_1$

Therefore, identity $= \sigma\sigma^{-1} = \tau_1 \cdots \tau_{2k}\rho_{2m+1} \cdots \rho_1$. Then the identity is a product of an odd number of transposition.

Transpositions switch two elements.

The identity: In any form, if I switch i and j with a transposition, I have to eventually switch them back.

So the identity can only be written in terms of an even # of transpositions.

1(a) 4×4 Latin Squares.

(1) →				
(2) →				

Ex:

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

(1) The first row corresponds to all permutations of the set $X = \{1,2,3,4\}$ and there 24 possibilities.

(2) The second row, we cannot have two number that are same in the column: Take any permutation in the first row

2	1	4	3

a	b	c	d

The permutation I can put in the second row cannot fix any letter in the first (Derangement)

$$d(4) = 4! \sum_{i=0}^{4} \frac{(-1)^i}{i!} = 4!\left(\cancel{\frac{1}{0!}} \cancel{- \frac{1}{1!}} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$
$$= 12 - 4 + 1 = 9$$

# of possibilities in the second row is 9.

Derangement is a permutation which does not fix any value.

Derangement vs. Not a derangement.

Ex: S_4: $(12)(34)$: $1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 4$, $4 \mapsto 3$ vs. (123): $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$, $4 \mapsto 4$

(3) Depends on the derangement chosen.

S_4: "double transpositions"
"4-cycles"

Double Transposition.

(12)(34)
(13)(24)
(14)(23)

6 4-cycles

(1234) (1423)
(1243) (1432)
(1324)
(1342)

(1 2)(3 4)

b	a	d	c
a	b	c	d

(1 2 3 4)

b	a	d	c
c	b	a	d

Possible 3rd Rows:

3 4 1 2
4 3 1 2
3 4 2 1
4 3 2 1

3 2 1 4
4 3 2 1

$24 \times 3 \times 4 \quad + \quad 24 \times 6 \times 2$

576 possible Latin squares.