Monday Section (5/10/21)

For recursive sequences, we have the following result for solutions:

Theorem: If $f(n) + \alpha f(n-1) + \beta f(n-2) = 0$. Then, the "characteristic polynomial" is

$$ch(t) = t^2 + \alpha t + \beta$$

If r_1 and r_2 are distinct roots of $ch(t)$, then the solution to the recurrence relation is

$$f(n) = c_1 r_1^n + c_2 r_2^n$$

where c_1 and c_2 are constants determined by $f(0)$ and $f(1)$.

If $ch(t)$ has a repeated root r, then the solution to the recurrence relation is

$$f(n) = c_1 r^n + c_2 \cdot n \cdot r^n$$

where c_1 and c_2 are constants determined by $f(0)$ and $f(1)$.

In the case of quadratic characteristic polynomials, it is enough to check:

For $ch(t) = at^2 + bt + c$,
- if $b^2 - 4ac \neq 0$, then $ch(t)$ has two unique roots
- if $b^2 - 4ac = 0$, then $ch(t)$ has only one repeated root.

This follows from:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For characteristic polynomials of higher degree, this does not work.

Example: Let $f(n) = 4f(n-1) + 4f(n-2) + 5f(n-3)$. $f(0) = 1$, $f(1) = 3$, $f(2) = 5$

Assume $f(n) = t^n$. Then,

$$t^n = 4t^{n-1} + 4t^{n-2} + 5t^{n-3}$$
$$t^n - 4t^{n-1} - 4t^{n-2} - 5t^{n-3} = 0$$
$$t^{n-3}(t^3 - 4t^2 - 4t - 5) = 0$$

Then, $ch(t) = t^3 - 4t^2 - 4t - 5$. There are 3 possibilities:

(1) $ch(t) = (t-r)^3$ Solution: $f(n) = c_1 r^n + c_2 \cdot n \cdot r^n + c_3 \cdot n^2 \cdot r^n$

(2) $ch(t) = (t-r_1)^2(t-r_2)$ Solution: $f(n) = c_1 r_1^n + c_2 n r_1^n + c_3 r_2^n$.

(3) $ch(t) = (t-r_1)(t-r_2)(t-r_3)$ Solution: $f(n) = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n$

The solution are dependent on the number of unique roots (and multiplicity).

Observations:

- Any cubic polynomial has a real root.

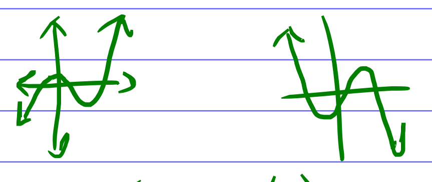
  $p(x) = ax^3 + bx^2 + cx + d$

  if $a > 0$, $\lim\limits_{x \to \infty} p(x) = \infty$ and $\lim\limits_{x \to -\infty} p(x) = -\infty$

  if $a < 0$, $\lim\limits_{x \to \infty} p(x) = -\infty$ and $\lim\limits_{x \to -\infty} p(x) = \infty$

  By continuity of $p(x)$, for all $-\infty < y < \infty$, there exists $x \in \mathbb{R}$ such that $f(x) = y$. (In particular, $y = 0$)

  Visually:



- Complex roots come in (conjugate) pairs.

  Suppose not, then

  $p(x) = (x + u + iv)(ax^2 + bx + c)$ $(v \neq 0)$

  $= ax^3 + bx^2 + cx + uax^2 + ubx + uc + ivax^2 + ivbx + ivc$

  $= ax^3 + (b + ua + iva)x^2 + (c + ub + ivb)x + ub + ivc.$

  $a = 0$ $b = 0$ $c = 0$

This implies $p(x) = 0$, but $p(x)$ is cubic.

(1) $ch(t) = (t-r)^3$ Solution: $f(n) = c_1 r^n + c_2 \cdot n \cdot r^n + c_3 \cdot n^2 \cdot r^n$

(2) $ch(t) = (t-r_1)^2 (t-r_2)$ Solution: $f(n) = c_1 r_1^n + c_2 n r_1^n + c_3 r_2^n$.

(3) $ch(t) = (t-r_1)(t-r_2)(t-r_3)$ Solution: $f(n) = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n$

(1) all real roots

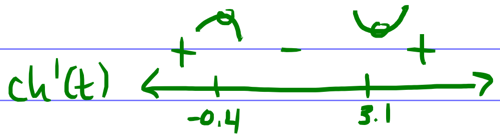
(2) all real roots

(3) all real roots or two complex, one real.

$ch(t) = t^3 - 4t^2 - 4t - 5$

$ch'(t) = 3t^2 - 8t - 4$

$t = \frac{8 \pm \sqrt{112}}{6} \approx 3.1$ or -0.4



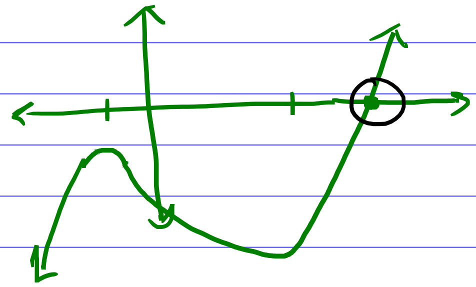
$ch'(4) = 12 > 0$

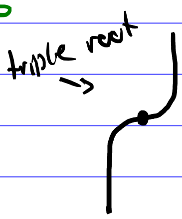
$ch'(0) = -4 < 0$

$ch'(-1) = 7 > 0$

$ch\left(\frac{8-\sqrt{112}}{6}\right) \approx -4.1$

$ch\left(\frac{8+\sqrt{112}}{6}\right) \approx -26$





Unique roots: General solution: $f(n) = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n$

$r_1 = 5$, $r_2 = -\sqrt[3]{-1}$, $r_3 = \sqrt[3]{-1}^{\,2}$

Thus, $f(n) = c_1 \cdot 5^n + c_2 (-\sqrt[3]{-1})^n + c_3 (\sqrt[3]{-1}^{\,2})^n$

15. (a) In an election, there are two candidates, A and B; the number of votes cast is $2n$. Each candidate receives exactly n votes; but, at every intermediate point during the count, A has received more votes than B. Show that the number of ways this can happen is the Catalan number C_n. [HINT: A leads by just one vote after the first vote is counted. Suppose that this next occurs after $2i+1$ votes have been counted. Then there are $f(i)$ choices for the count between these points, and $f(n-i)$ choices for the rest of the count, where $f(n)$ is the required number; so we obtain the Catalan recurrence.]

$C_0 = 1, \ C_1 = 1$

$C_2 = 2, \ C_3 = 5$

The author should have "the number of votes cast is $2n+2$". To see why:

When 2 votes casted:

AB $\qquad C_0 = 1$ possibility

When 4 votes casted

AABB $\qquad C_1 = 1$ possibility

When 6 votes casted

AAABBB

AABABB $\qquad C_2 = 2$ possibility

HW 5:

2(a) $F_n^2 - F_{n+1}F_{n-1} = (-1)^n$ for $n \geq 1$ $\qquad F_n = F_{n-1} + F_{n-2}$

Base Case: $n=1$. Plug in.

Assume that the formula holds for $n \leq k$

$$F_{k+1}^2 - F_{k+2}F_k$$

Goal: To find $\boxed{F_k^2 - F_{k+1}F_{k-1}}$ $\qquad F_n = F_{n-1} + F_{n-2}$

$F_n - F_{n-1} = F_{n-2}$

$$F_{k+1}^2 - F_{k+2}F_k$$

① $(F_k + F_{k-1})^2 - F_{k+2}F_k$

$F_k^2 + 2F_kF_{k-1} + \underbrace{F_{k-1}^2} - F_{k+2}F_k$

or

② $F_{k+1}^2 - (F_{k+1} + F_k)F_k$

$F_{k+1}^2 - F_{k+1}F_k - F_k^2$

② $F_{k+1}^2 - (F_{k+1} + F_k)F_k$

$F_{k+1}^2 - F_{k+1}F_k - F_k^2$

$F_{k+1}\underbrace{(F_{k+1} - F_k)} - F_k^2$ $\qquad F_{k+1}^2 - F_k(\underbrace{F_{k+1} + F_k}_{= F_{k+2}})$

9(ii) $f(n+1) = f(n) + f(n-1) + f(n-2)$ $\quad f(0) = f(1) = f(2) = 1$

- characteristic polynomial

(1) $ch(t) = (t-r)^3$ Solution: $f(n) = c_1 r^n + c_2 \cdot n \cdot r^n + c_3 \cdot n^2 \cdot r^n$

(2) $ch(t) = (t-r_1)^2 (t-r_2)$ Solution: $f(n) = c_1 r_1^n + c_2 n r_1^n + c_3 r_2^n$.

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- How does factor into roots
  (three the same)
  (two different, one repeated)
  (all three different)

Steps: - Subsititute $f(n) = t^n$
- Find characteristic polynomial
- Determine how it factors (not necessarily finding the roots explicitly)
- Write the general equation accordingly
- (Don't have to solve explictly)