Section 2 Exercise 5(a)

Let $X = \{1,2,3\}$.

Consider the power set:

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}$$

Recall that two unlabelled families of subsets are the same if there exists a permutation which sends the labels of one family to the other.

Ex: $\{\{1\},\{1,2\}\} \cong \{\{2\},\{2,3\}\}$ as unlabelled sets since the permutation $\sigma: X \to X$ is such that $\{\{\sigma(1)\}, \{\sigma(1),\sigma(2)\}\} = \{\{2\},\{2,3\}\}$.

$1 \mapsto 2$
$2 \mapsto 3$
$3 \mapsto 1$

First, let's simplify the problem. Note $\emptyset$ and X are unaffected by permutations. There is nothing to permute in $\emptyset$ and σ is a bijection on X, so permuting yields the same set. For any given family of subsets $\mathcal{F}$, it can either contain $\emptyset$ or not and can either contain X or not. Therefore, we are left with finding how many configurations there are of 1-sets and 2-sets (we can then multiply the number of configurations by 4 to get total number of unlabeled families).

Start with no 1-subset. Then we have

zero 2-sets	$\emptyset$	configuration 1
one 2-set	$\{\{1,2\}\} \cong \{\{1,3\}\} \cong \{\{2,3\}\}$	configuration 2
two 2-sets	$\{\{1,2\},\{2,3\}\} \cong \{\{1,2\},\{1,3\}\} \cong \{\{1,3\},\{2,3\}\}$	configuration 3
three 2-sets	$\{\{1,2\},\{1,3\},\{2,3\}\}$	configuration 4

If we have one 1-set.

zero 2-sets	$\{\{1\}\} \cong \{\{2\}\} \cong \{\{3\}\}$	configuration 1
one 2-set	$\{\{1\},\{1,2\}\} \cong \{\{1\},\{1,3\}\} \cong \{\{2\},\{2,1\}\} \cong \{\{2\},\{2,3\}\} \cong \{\{3\},\{1,3\}\} \cong \{\{3\},\{2,3\}\}$	configuration 2
	$\{\{1\},\{2,3\}\} \cong \{\{2\},\{1,3\}\} \cong \{\{3\},\{1,2\}\}$	configuration 3

Key Observation: We have two possibilities for one 1-set and one 2-set.

- When the 1-set and 2-set <u>share</u> an element.
- When the 1-set and 2-set <u>DON'T</u> <u>share</u> an element.

two 2-sets: By a similar argument, we get two possibilities when there is one 1-set and two 2-sets. (2 possibilities) configuration 4 and configuration 5.

three 2-sets: $\{\{1\},\{1,2\},\{1,3\},\{2,3\}\} \cong \{\{1\},\{1,2\},\{1,3\},\{2,3\}\} \cong \{\{1\},\{1,2\},\{1,3\},\{2,3$

This gives a total of 6 possibilities when we have one 1-set.

Overall, we can make an argument that:

$$\left.\begin{array}{l} 0 \text{ 1-sets} \longrightarrow 4 \text{ possibilities} \\ 1 \text{ 1-sets} \longrightarrow 6 \text{ possibilities} \\ 2 \text{ 1-sets} \longrightarrow 6 \text{ possibilities} \\ 3 \text{ 1-sets} \longrightarrow 4 \text{ possibilities} \end{array}\right\} \text{20 possible configurations.}$$

We then multiply by 4 for whether or not we add in $\emptyset$ or X.

Thus <u>80</u> unlabeled families of subsets of $X = \{1,2,3\}$.