Wednesday Section (4/28/21)

Definitions:

- A **partial preorder** P is a reflexive and transitive relation
- A **preorder** P is a reflexive and transitive relation such that for any $x, y \in P$, exactly one condition is satisfied:
  (1) $(x,y) \in P$
  (2) $(y,x) \in P$ (trichotomy)
  (3) $x = y$
- A **partial order** P is a reflexive, transitive, and antisymmetric relation.
- A **order** P is a reflexive, transitive, and antisymmetric relation such that for any $x, y \in P$, exactly one condition is satisfied:
  (1) $(x,y) \in P$
  (2) $(y,x) \in P$ (trichotomy)
  (3) $x = y$

In other words,
Order satisfies reflexive, transitive, antisymmetric and trichotomy.
The prefix pre- removes the antisymmetric requirement.
The adjective partial removes the trichotomy requirement.

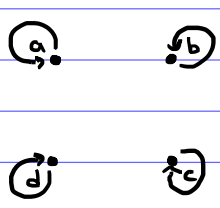
↳ Symmetric: If aRb, then bRa. $\boxed{aRa}$
Antisymmetric: If aRb and bRa, then $a=b$
Asymmetric: If aRb, but $b\not R a$.
Asymmetry is in negation of symmetry.

Let $X = \{a, b, c, d\}$. How many partial preorders exist on X?
Note: Every partial order P is reflexive.

$$\{(a,a),(b,b),(c,c),(d,d)\} \subseteq P$$

or is a subgraph of the (directed) graph associated to P.

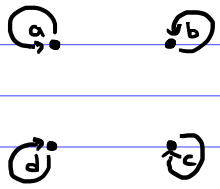
Approach: Classify by the number edges we add to



Note that once we have all partial preorders, we can find all preorders, partial orders, and orders by selecting all partial preorders that satisfy the additional conditions.

0 - edges added:

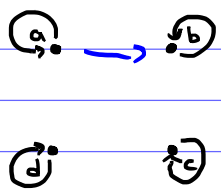
$P = \{(a,a), (b,b), (c,c), (d,d)\}$



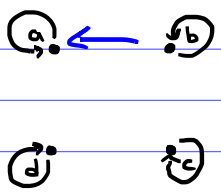
trivially transitive

1 - edge added

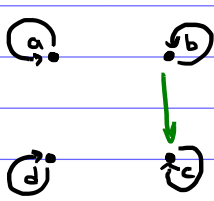
$P = \{(a,a), (b,b), (c,c), (d,d), (a,b)\}$



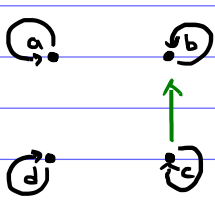
$P = \{(a,a), (b,b), (c,c), (d,d), (b,a)\}$



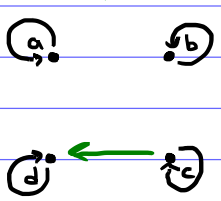
$P = \{(a,a), (b,b), (c,c), (d,d), (b,c)\}$



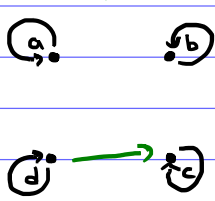
$P = \{(a,a), (b,b), (c,c), (d,d), (c,b)\}$



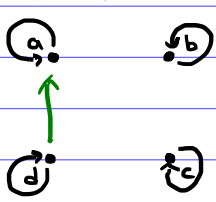
$P = \{(a,a), (b,b), (c,c), (d,d), (c,d)\}$



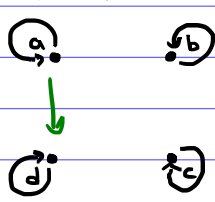
$P = \{(a,a), (b,b), (c,c), (d,d), (d,c)\}$



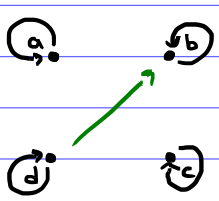
$P = \{(a,a), (b,b), (c,c), (d,d), (d,a)\}$



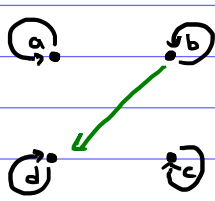
$P = \{(a,a), (b,b), (c,c), (d,d), (a,d)\}$



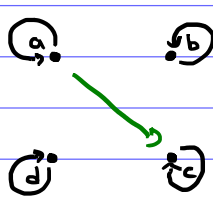
$P = \{(a,a), (b,b), (c,c), (d,d), (d,b)\}$



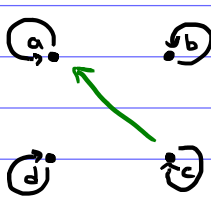
$P = \{(a,a), (b,b), (c,c), (d,d), (b,d)\}$



$P = \{(a,a), (b,b), (c,c), (d,d), (a,c)\}$



$P = \{(a,a), (b,b), (c,c), (d,d), (c,a)\}$

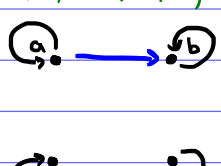


2-edges added: Consider fixing a particular edge and finding all partial preorders which contain that edge.

Fix $(a,b) \in P$

Consider $(a,x) \in P$

$P = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,\ )\}$



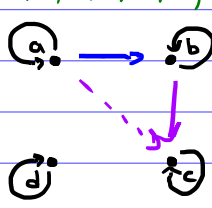
Either $(a,c) \in P$ or $(a,d) \in P$ doesn't contradict transitivity

$P = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c)\}$

$P = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,d)\}$

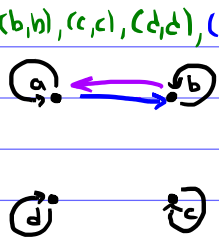
Consider $(b,x) \in P$

$P = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,x)\}$



If $x=c$ or $x=d$, then not transitive.

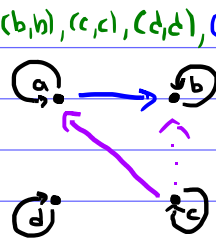
$P = \{(a,a),(b,b),(c,c),(d,d),(\ ,\ ),(\ ,\ )\}$



This is transitive (symmetric)

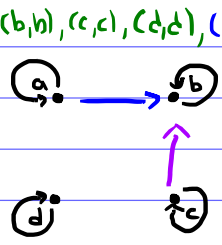
Consider $(c,x) \in P$

$P = \{(a,a),(b,b),(c,c),(d,d),(\ ,\ ),(\ ,\ )\}$



Not transitive

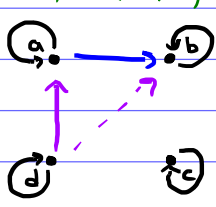
$P = \{(a,a),(b,b),(c,c),(d,d),(\ ,\ ),(\ ,\ )\}$



Either $(c,b) \in P$ or $(c,d) \in P$ gives us a partial preorder.

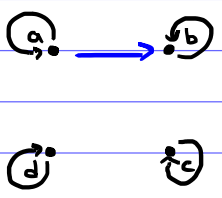
Consider $(d,x) \in P$

$P = \{(a,a),(b,b),(c,c),(d,d),(\ ,\ ),(\ ,\ )\}$



Not transitive.

$P = \{(a,a),(b,b),(c,c),(d,d),(\ ,\ ),(\ ,\ )\}$



Either $(d,b) \in P$ or $(d,c) \in P$ gives us a partial preorder.

Observe: Fixed $(a,b) \in P$

- We could choose any $(a,x) \in P$ (2)
- ✱ We could only choose $(b,a) \in P$ (1)
- We could choose (c,x) and (d,x) where $x \neq a$. (4)

Note if we fix $(b,a) \in P$, then

- We could choose any $(b,x) \in P$ (2)
- ✱ We could choose $(a,b) \in P$ (1)
- We could choose (c,x) and (d,x) where $x \neq b$ (4)

✱ These are the same relation. So 13 possible that include (a,b) or (b,a).

We can now split this into cases:

- Partial preorders in which $(a,b) \in P$ or $(b,a) \in P$. (13)
- Partial preorders in which $(a,b) \notin P$ and $(b,a) \notin P$ (?)

Then, we can continue the pattern...

The main takeaway: Try to create an algorithm that goes through all possibilities.

What about preorders, partial orders and orders?

For preorders and orders, we have trichotomy:

$X = \{a, b, c, d\}$

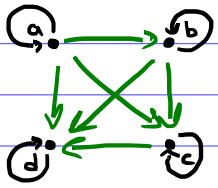
For any $x, y \in X$, exactly one condition is satisfied:

(1) $(x, y) \in P$

(2) $(y, x) \in P$

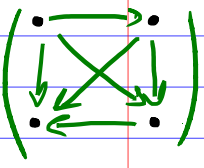
(3) $x = y$

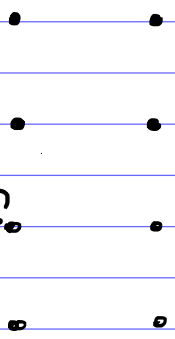
What does this translate to graphically?



There must be a (directed) edge between each pair of vertices.

A trichotomy is an antisymmetric and "complete" graph.

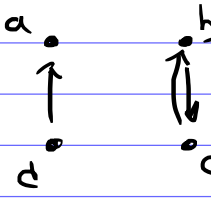




antisymmetric, symmetric, transitive, irreflexive.

antisymmetric, symmetric, transitive

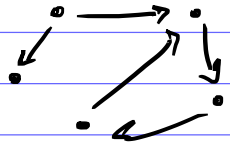
Antisymmetric is not the opposite of symmetric.



Neither antisymmetric or symmetric

$a R d$ but $d \not R a$ — not symmetric

$b R c$ and $c R b$ but $b \neq c$ — not antisymmetric.



Bell number is the number of partitions of a n-set.

$B_1 = 1$ $\{1\}$. $\{\{1\}\}$

$B_2 = 2$ $\{1,2\}$, $\{\{1,2\}\}$ or $\{\{1\},\{2\}\}$

$B_3 = 5$ $\{1,2,3\}$ $\{\{1,2,3\}\}$ or $\{\{1\},\{2,3\}\}$ or $\{\{1\},\{2\},\{3\}\}$
$\{\{2\},\{1,3\}\}$
$\{\{3\},\{2,1\}\}$

1

1 2

2 3 5

5 7 10 15

15 20 27 37 52

15 partitions of a 4-set.

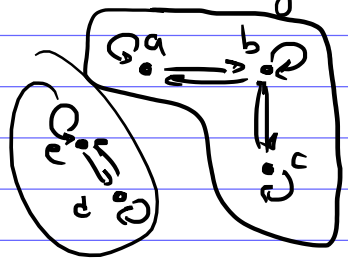
52 partitions of a 5-set.

Partition of a set X is a collection of nonempty sets $S_1, \ldots, S_n$ such that

$S_i \cap S_j = \emptyset$

$S_1 \cup \cdots \cup S_n = X$

An equivalence relation:
reflexive, symmetric and transitive.



$\{a,b,c\} \cup \{d,e\} = X$

12 people
pair together people into 6 groups.

ABCDEFGHIJKL

{B,E} {A,L}, {K,J}, { }, { }, { }

$$\left(\frac{12\cdot 11}{2!}\cdot\frac{10\cdot 9}{2!}\cdot\frac{8\cdot 7}{2!}\cdot\frac{6\cdot 5}{2!}\cdot\frac{4\cdot 3}{2!}\cdot\frac{2\cdot 1}{2!}\right)/6!$$

11

$$\frac{\binom{12}{2}\cdot\binom{10}{2}\cdot\binom{8}{2}\cdot\binom{6}{2}\cdot\binom{4}{2}\cdot\binom{2}{2}}{6!}$$

$2\cdot 4\cdot 3$

ABCD $\frac{\binom{4}{2}\binom{2}{2}}{2!} = \frac{\frac{4!}{2!2!}\cdot\frac{2!}{2!0!}}{2!}$ 3

{A,B}, {C,D}
{A,C}, {B,D}
{A,D}, {B,C}

{C,D}, {A,B}
{B,D}, {A,C}
{B,C}, {A,D}

we only get these since the order in which we choose groups does not matter

4-digit pin 0123456789

No digit is repeated, but 3 and 9 must be present.

8·7

39__
3_9_
3__9
_39_
_3_9
__39

2x

8·7·6·2

$\frac{8!}{6!2!}\cdot 4!$

$\frac{8\cdot 7}{2!}\cdot 4\cdot 3\cdot 2\cdot 1$

$8\cdot 7\cdot 6\cdot 2$

{3,9,_,_} ← $\binom{8}{2}$

(3,9,_,_) $\binom{8}{2}\cdot 4!$

11. How many non-negative integer solutions to

$$x_1 + \dots + x_4 = k$$

$X = \{a, b, c, d, e\}$ select 3 things from X

	ordered	unordered
repetition	5^3	$\binom{5+3-1}{3} = \binom{7}{3}$
no repetition	$\frac{5!}{(5-3)!}$	$\binom{5}{3}$

$\frac{7!}{4!3!}$

$\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1}$

3

35

$$x_1 + x_2 + x_3 + x_4 + x_5 = 3$$

$x_i \geq 0$

(a) $x_1 + \dots + x_n = k$ $\quad \binom{n+k-1}{k}$

(b) $x_1 + x_2 + x_3 + x_4 = 20$

$1 \leq x_1 \iff 0 \leq x_1 - 1 = y_1$ ← non-negative integer

$2 \leq x_2$

$3 \leq x_3$

$4 \leq x_4$

$x_1 - 1 + x_2 + x_3 + x_4 = 20 - 1$

$y_1 + x_2 + x_3 + x_4 = 19$

$y_1 + x_2 - 2 + x_3 - 3 + x_4 - 4 = 19 - 2 - 3 - 4$

$y_1 + y_2 + y_3 + y_4 = 10$

$n = 4$ $\quad k = 10$

$$\binom{4+10-1}{10}$$

$x_1 + x_2 + x_3 + x_4 = 20$

$(x_1 - 1) + 1 + (x_2 - 2) + 2 + (x_3 - 3) + 3 + (x_4 - 4) + 4 = 20$

$(x_1 - 1) + (x_2 - 2) + (x_3 - 3) + (x_4 - 4) = 20 - 1 - 2 - 3 - 4$

$= 10$

$-1 \leq x_1$

$f(x) = x^3 + 2x$ $\quad t = x + 1$

$f(t) = (t-1)^3 + 2(t-1)$ $\quad t - 1 = x$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} = 2^{n-1}$$

$$\sum_{k=0}^{\lfloor \frac{1}{2} \rfloor} \binom{1}{2k} = \sum_{k=0}^{0} \binom{1}{2k} = \binom{1}{0} = 1$$

Base case:

$$\sum_{k=0}^{\lfloor \frac{2}{2} \rfloor} \binom{2}{2k} = \binom{2}{0} + \binom{2}{2} = 2$$

Assume $\lfloor \frac{n}{2} \rfloor$

$$\sum_{k=0} \binom{n}{2k} = 2^{n-1}$$

Show $\lfloor \frac{n+1}{2} \rfloor$

$$\sum_{k=0} \binom{n+1}{2k} = 2^n$$ ✗

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

$$\sum_{k=0}^{n} \left[\binom{n}{k} + (-1)^k \binom{n}{k} \right] = 2^n$$

If k is odd

$$\binom{n}{k} - \binom{n}{k} = 0$$

If k is even

$$\binom{n}{k} + \binom{n}{k} = 2\binom{n}{k}$$

Let $k = 2i$.

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \left[2\binom{n}{2i} \right] = 2 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} = 2^n$$

$$\sum_{k=0}^{4} \left[\binom{n}{k} + (-1)^k \binom{n}{k} \right] = 2^4$$

$$\left[\binom{n}{0} + (-1)^0 \binom{n}{0} \right] + \left[\binom{n}{1} + (-1)^1 \binom{n}{1} \right] + \left[\binom{n}{2} + (-1)^2 \binom{n}{2} \right] + \left[\binom{n}{3} + (-1)^3 \binom{n}{3} \right] + \left[\binom{n}{4} + (-1)^4 \binom{n}{4} \right]$$

$$2\binom{n}{0} + 0 + 2\binom{n}{2} + 0 + 2\binom{n}{4}$$

$$\sum_{i=0}^{2} 2\binom{n}{2i} = 2^4$$

$$2\left(\sum_{i=0}^{2} \binom{n}{2i} \right) = 2^4$$

$$\sum_{i=0}^{2} \binom{n}{2i} = 2^3$$

12b. $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

$2 \cdot \binom{4}{2}$

Fact: $k\binom{n}{k} = n\binom{n-1}{k-1}$

$$\sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} n\binom{n-1}{k-1} = n\left(\sum_{k=1}^{n} \binom{n-1}{k-1} \right)$$

$$= n\left(\binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{n-1} \right) = n \sum_{k=0}^{n-1} \binom{n}{k} = n2^{n-1}$$

29 preorders on a set of 3 elements

$B_n < n!$ Hint: assign it a permutation.

f

$\{1,2,3\} \longrightarrow (123)$ or (132)

$\{1,2\}\{3\} \longrightarrow (12)$

$\{1,3\}, \{2\} \longrightarrow (13)$

$\{2,3\}, \{1\}$ (23)

$\{1\}, \{2\}, \{3\}$ identity

$f(\text{Partitions}) \leq S_n$.

14. $\mathbb{Z}^\times = \mathbb{Z} \setminus \{0\}$

Show $R = \{(m,n) : m,n \in \mathbb{Z}^\times \text{ and } mn > 0\}$

$(1,3)$ $\quad 1 \cdot 3 > 0$

$(-2,-5)$ $\quad -2 \cdot -5 = 10 > 0$

$(-3,+2)$ $\quad -3 \cdot 2 < 0$

Reflexive: $x^2 > 0$ so $(x,x) \in R$

Symmetry: if $xy > 0$, then $yx > 0$

if $(x,y) \in R$, then $(y,x) \in R$.

Transitive: $xy > 0$ and $yz > 0$

if x is negative, so is y.

since y is negative, then z negative.

$yz > 0$

$\mathbb{Z}^\times = \{1,2,3,4,\ldots\} \cup \{-1,-2,-3,\ldots\}$

$S_1 \qquad\qquad S_2$

If $x \in S_1$ and $x \in S_2$, then for any $y \in S_1$

yRx

for any $z \in S_2$

xRz

transitivity yRz. $S_1 = S_2$

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