Monday Section (4/26/21)

Definitions:

- A partial preorder P is a reflexive and transitive relation
- A preorder P is a reflexive and transitive relation such that for all $x, y \in P$, exactly one condition is satisfied:
  (1) $(x,y) \in P$
  (2) $(y,x) \in P$ (trichotomy)
  (3) $x = y$
- A partial order P is a reflexive, transitive, and antisymmetric relation.
- A order P is a reflexive, transitive, and antisymmetric relation such that for all $x, y \in P$, exactly one condition is satisfied:
  (1) $(x,y) \in P$
  (2) $(y,x) \in P$ (trichotomy)
  (3) $x = y$

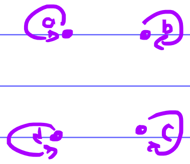
In other words,

Order satisfies reflexive, transitive, antisymmetric and trichotomy.
The prefix pre- removes the antisymmetric requirement.
The adjective partial removes the trichotomy requirement.

Let $X = \{a, b, c, d\}$. How many partial preorders exist on X?
Note: Every partial order P is reflexive, thus:

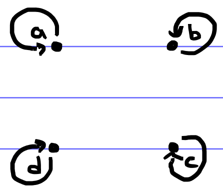
$$\{(a,a), (b,b), (c,c), (d,d)\} \subseteq P$$

or



is a subgraph of the (directed) graph associated to P.

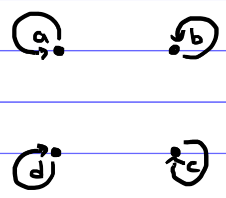
Approach: Classify by the number edges we add to



Note that once we have all partial preorders, we can find all preorders, partial orders, and orders by selecting all partial preorders that satisfy the additional conditions.

0-edges added:

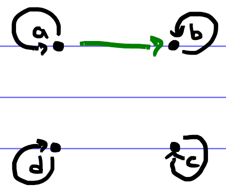
$P=\{(a,a),(b,b),(c,c),(d,d)\}$



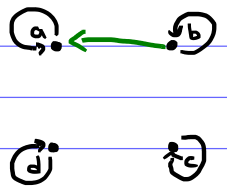
trivially transitive

1-edge added

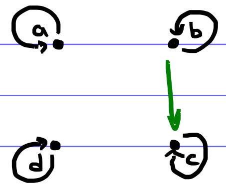
$P=\{(a,a),(b,b),(c,c),(d,d),(a,b)\}$



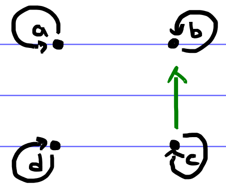
$P=\{(a,a),(b,b),(c,c),(d,d),(b,a)\}$



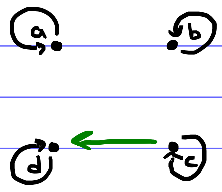
$P=\{(a,a),(b,b),(c,c),(d,d),(b,c)\}$



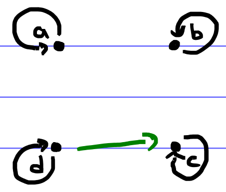
$P=\{(a,a),(b,b),(c,c),(d,d),(c,b)\}$



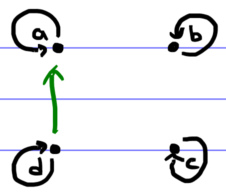
$P=\{(a,a),(b,b),(c,c),(d,d),(c,d)\}$



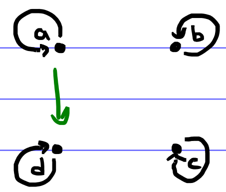
$P=\{(a,a),(b,b),(c,c),(d,d),(d,c)\}$



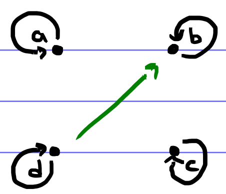
$P=\{(a,a),(b,b),(c,c),(d,d),(d,a)\}$



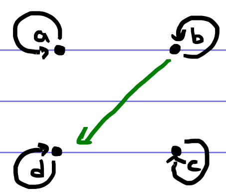
$P=\{(a,a),(b,b),(c,c),(d,d),(a,d)\}$



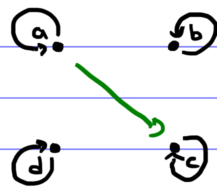
$P=\{(a,a),(b,b),(c,c),(d,d),(d,b)\}$



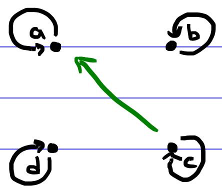
$P=\{(a,a),(b,b),(c,c),(d,d),(b,d)\}$



$P=\{(a,a),(b,b),(c,c),(d,d),(a,c)\}$



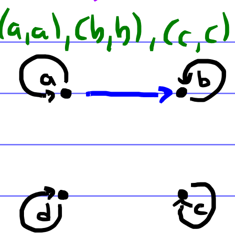
$P=\{(a,a),(b,b),(c,c),(d,d),(c,a)\}$



2-edges added: Instead of listing them all, we can start with fixing $(a,b) \in P$

Consider $(a, x) \in P$

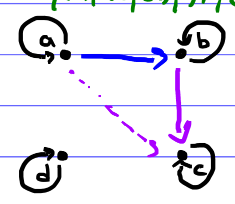
$P = \{(a,a), (b,b), (c,c), (d,d), (a,b), (a, \ )\}$



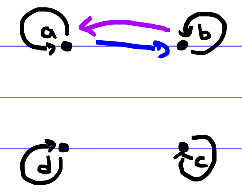
Either adding (a,c) or (a,d) in the relation doesn't transitivity

Consider $(b, x) \in P$

$P = \{(a,a), (b,b), (c,c), (d,d), (a,b), (b, \ )\}$

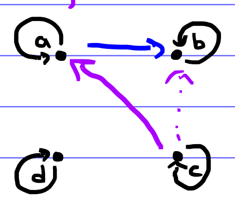


This won't be transitive. (This is not a partial preorder)

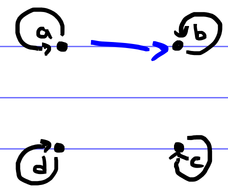


This is transitive (and symmetric).

Consider $(c, x) \in P$

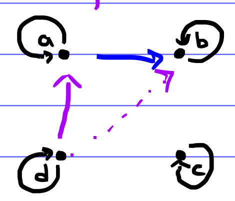


This won't be transitive.

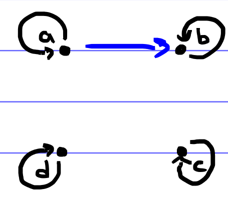


Either (c,b) or (c,d) is transitive.

Consider $(d, x) \in P$



This won't be transitive



Either (d,c) or (d,b).

Observe:

- we could choose any $(a,x) \in P$ (2)
- ☆ - we could only choose $(b,a) \in P$ (1)
- we could choose (c,x) and (d,x) where $x \neq a$. (4)

Note if we fix (b,a), then

- we could choose any $(b,x) \in P$ (2)
- ☆ - we could choose only $(a,b) \in P$ (1)
- we could choose (c,x) and (d,x) where $x \neq b$ (4)

☆ These are the same relation, so we have: 13 partial preorders in which $(a,b) \in P$ ~~and~~ or $(b,a) \in P$

We can now split this into cases:

- Partial preorders in which $(a,b) \in P$ or $(b,a) \in P$.
- Partial preorders in which $(a,b) \notin P$ and $(b,a) \notin P$

Then, we can continue the pattern...

The main takeaway: Try to create an algorithm that goes through all possibilities.

What about preorders, partial orders and orders?

For preorders and orders, we have trichotomy:

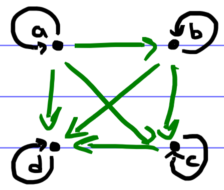
For any $x, y \in P$, exactly one condition is satisfied:

(1) $(x,y) \in P$

(2) $(y,x) \in P$

(3) $x = y$

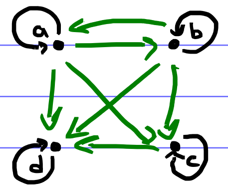
What does this translate to graphically?



There must be a (directed) edge between each pair of vertices.

$R = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(a,d),(b,c)(b,d),(c,d)\}$

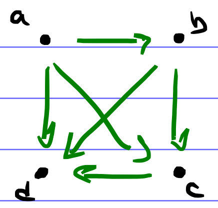
If we have a (partial) preorder, we can also have directed edges in the other direction. For example:



However, if we have an order, then we must have exactly one directed edge between every pair of vertices.

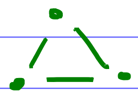
Antisymmetric
at most one directed edge between two vertices

Trichotomy
exactly one directed edge between two vertices



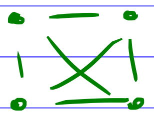
Trichotomy

Graphs on 3 vertices



$R = \{(a,b),(b,c)\}$

Graphs on 4 vertices:



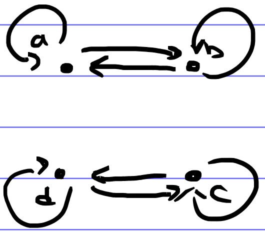
$|R| < 6$ it is not a trichotomy.

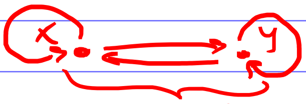
$|R| \geq 6$ (it could or could not be a trichotomy)

$A = \{a,b,c,d\}$

$R1 = \{(a,a),(b,a),(b,b),(c,c),(c,d),$
$(d,c),(d,d),(a,b)\}$

An equivalence is reflexive
symmetric
transitive



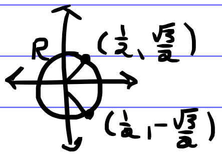


this not transitive

xRy and yRx
then xRx

$R_1(a) = \{$ the set of outputs of $a\}$.
$= \{a,b\}$

$R_1(b) = \{a,b\}$



$R1 = \{(a,a),(b,a),(b,b),(c,c),(c,d),$
$(d,c),(d,d),(a,b)\}$

$R(\frac{1}{2}) = \{\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\}$

$R_1(a) = \{a,b\}$ if and only if $\{(a,a),(a,b)\} \subseteq R_1$

$$\begin{array}{ll} & A \\ & \parallel \\ R_1(a) = \{a,b\} = R_1(b) \longleftrightarrow & \{(a,a),(a,b),(b,a),(b,b)\} \subseteq R_1 \\ R_1(c) = \{c,d\} = R_1(d) \longleftrightarrow & \{(c,c),(c,d),(d,c),(d,d)\} \subseteq R_1 \\ & \parallel \\ & B \end{array}$$

$R_2 = \{(a,a),(a,b),(b,b),(c,c),(c,d)\}$

$R_2(a) = \{a,b\}$ $\qquad R_2(a) \cap R_2(c) = \emptyset$
$R_2(c) = \{c,d\}$ $\qquad R_2(a) \cup R_2(c) = A$

"$R_1(a)$ is the set of all $x \in A$ such that aR_1x."