Wednesday Section (4/21/21)

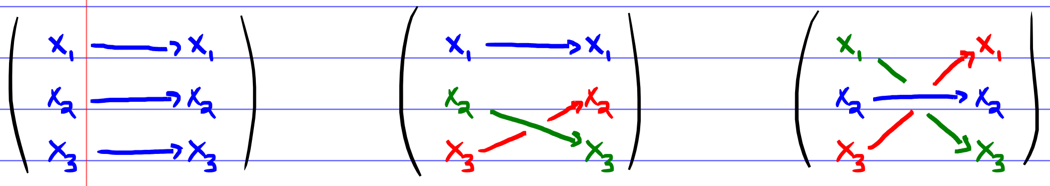
Recall from lectures:

There is a bijection between the number of ways arrange n objects (permutations) and the symmetric group of order n.

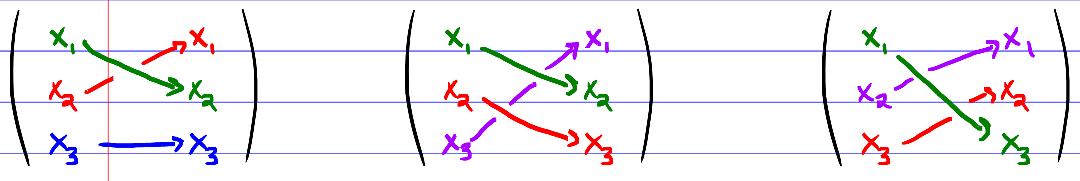
Example:

$$(23)(1) = 1$$
$$(23)(2) = 3$$
$$(23)(3) = 2$$

For $X = \{x_1, x_2, x_3\}$,



$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = e \quad \text{identity} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (23) \qquad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (13)$$



$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12) \qquad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123) \qquad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)$$

As $n \to \infty$, it becomes more difficult to write down all the elements explicitly. We can consider specific types of elements:

Transpositions or 2-cycles:

In S_2: (12) — 1

In S_3: (12); (13), (23) — $1+2$

In S_4: (12); (13), (23); (14), (24), (34) — $1+2+3$

In S_5: (12); (13), (23); (14), (24), (34); (15), (25), (35), (45) — $1+2+3+4$

How many transpositions are there in S_n?

From above, we know that

$$1+2+\cdots+(n-1) = \sum_{i=1}^{n-1} i = \frac{(n-1)n}{2} \qquad \left(\text{By } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right)$$

$$= \frac{n(n-1)(n-2)(n-3)\cdots 2\cdot 1}{2\cdot 1\ (n-2)(n-3)\cdots 2\cdot 1}$$

$$= \frac{n!}{2!(n-2)!} = \binom{n}{2}$$

So, a proof can involve selecting 2 elements from a set of n element, which does not depend on the order.

Sketch: Any transposition is of the form $(i\,j)$

We choose $i \in \{1, \ldots, n\}$

$j \in \{1, \ldots, n\}$ such that $j \neq i$

Note that $(i\,j) = (j\,i)$ for $i, j \in \{1, \ldots, n\}$ (for $j \neq i$)

Therefore, the total number of transposition is:

$$\frac{n\cdot(n-1)}{2!} = \binom{n}{2} \qquad \boxtimes$$

In group theory, this result gives insight into the structure of S_n.

Generalizing to k-cycles for $1 \le k \le n$ in S_n.

S_3 2-cycles: $(1\,2)$ $(1\,3)$
$(2\,3)$

cycles = 3-cycles: $(1\,2\,3)$ $(1\,3\,2)$

S_4 2-cycles: $(1\,2)$ $(1\,3)$ $(1\,4)$
$(2\,3)$ $(2\,4)$
$(3\,4)$

3-cycles: $(1\,2\,3) \neq (1\,3\,2)$
but $(1\,2\,3) = (3\,1\,2)$

Listing out them all is more difficult.
- Order matters
- Each element is not repeated.

(1) Since no element is repeated, we can think of this as selecting from a set.
(2) Order matters, so these are ordered subsets
(3) Some of these ordered subsets are the same.
(4) I won't say anymore, it'll spoil HW #8.

Double transpositions: $(i\,j)(k\,l)$
The composition of 2 transpositions which cannot be simplified into a 1, 2, 3 or 4-cycle.

Example: In S_4: All four different:

$$= (1\,2) \circ (3\,4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$(1\,3) \circ (2\,4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$(1\,4) \circ (2\,3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Two choices the same:

$$(1\,2)(1\,3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} = (1\,3\,2)$$

$(1\,2) \circ (1\,3)(1) = (1\,2)(3) = 3$
$(1\,2) \circ (1\,3)(2) = (1\,2)(2) = 1$
$(1\,2) \circ (1\,3)(3) = (1\,2)(1) = 2$
$(1\,2) \circ (1\,3)(4) = (1\,2)(4) = 4$

$$(i\,k)(i\,j) = (i\,j\,k)$$

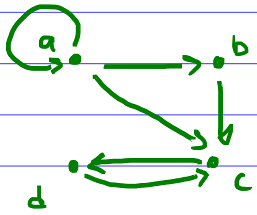
So, these are actually 3-cycles.
Since $(i\,j) = (j\,i)$ and $(i\,k) = (k\,i)$, none of these are double transpositions, since we consider double transpositions to be products which cannot be reduced to a different form.

Relations: A relation R on X is a subset of $X \times X$.

Ex: Let $X = \{a, b, c, d\}$.

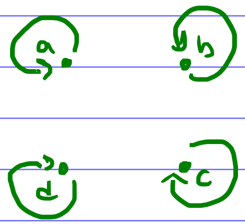
Let $R = \{(a,a), (c,d), (d,c), (a,b), (b,c), (a,c)\}$

We can associate a graph to the relation R.



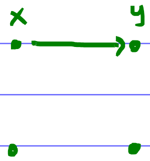
If xRy, then we draw an arrow from x to y.

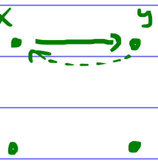
Ex: If R is a reflexive on X, then xRx for all $x \in X$, i.e.



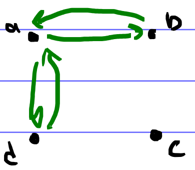
is a subgraph of the graph associated to the relation R.

Ex: If R is symmetric on X, then xRy implies yRx, i.e.

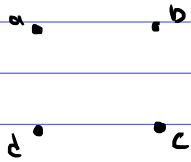
If 

, then 

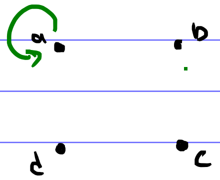
is a subgraph of the graph associated to R.

(1) 

$R = \{(a,b), (b,a), (a,d), (d,a)\}$

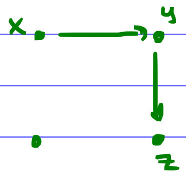
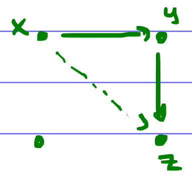
(2) 

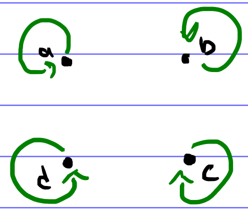
$R = \emptyset$.

(3) 

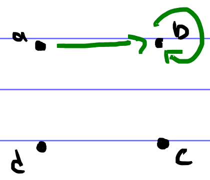
$R = \{(a,a)\}$.

Ex: If R is transitive on X, then if xRy and yRz, then xRz.

If  , then  is a subgraph of the graph associated to R.

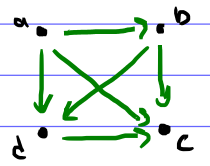
(1) 

$R = \{(a,a),(b,b),(c,c),(d,d)\}$

(2) 

$R = \{(a,b),(b,b)\}$

aRb and bRb, then aRb

(3) 

$R = \{(a,b),(a,c),(a,d),(b,c),(b,d),(d,c)\}$

In other words:

Reflexivity: Loops on each vertex

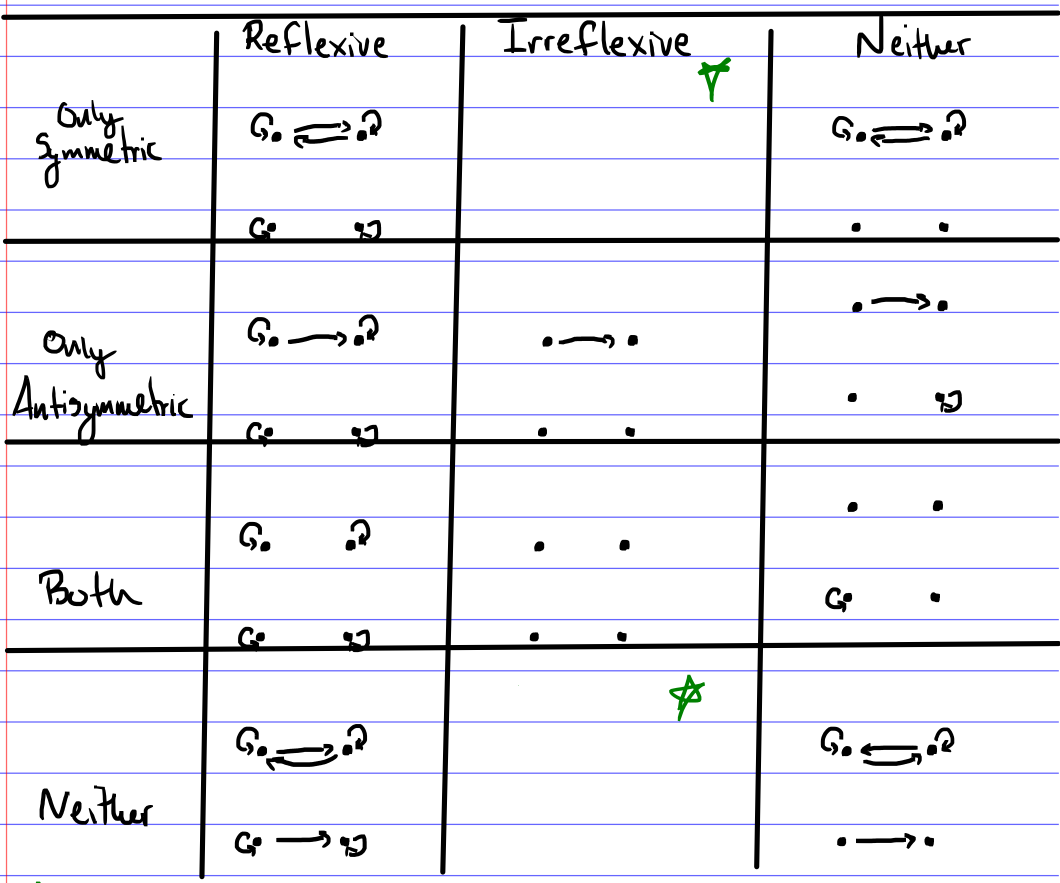
Symmetric: Arrows go in both direction (if one exist in one direction)

Transitivity: Able to compose arrows.
Pairs of arrows form direct triangles.

Irreflexivity: No loops on any vertex

Antisymmetric: Arrows between two distinct vertices go in only one direction.

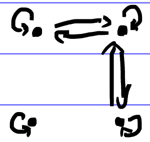
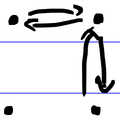
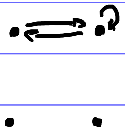
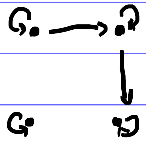
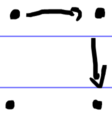
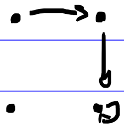
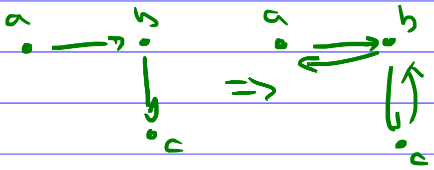
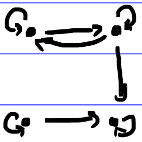
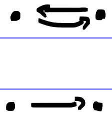
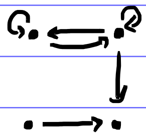
Transitive Relations:



✰ If a graph is symmetric, then a pair of arrows xRy and yRx implies xRx and yRy if R is transitive.

✰ The only symmetric and antisymmetric graph contain only loops.

Non-Transitive Relations:

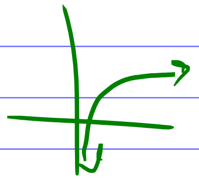
	Reflexive	Irreflexive	Neither
Only Symmetric			
Only Antisymmetric			
Both			$\Rightarrow a=b=c$ transitive
Neither			

$$F(n) \qquad f(n)$$
$$\uparrow \qquad\quad \uparrow$$
unlabelled $\quad$ labelled

$$\frac{f(n)}{n^n} \leq \frac{f(n)}{n!} \leq F(n) \leq f(n) \qquad f(n) = 2^{2^n}$$

$$n! \leq n^n \qquad\qquad n \log n = \log(n^n)$$
$$\frac{1}{n^n} \leq \frac{1}{n!}$$

$$\frac{2^{2^n}}{n^n} \leq \frac{2^{2^n}}{n!} \leq F(n) \leq 2^{2^n}$$



$$\log_2\left(\frac{2^{2^n}}{n^n}\right) \leq \log_2(F(n)) \leq \log_2(2^{2^n})$$

$$\log_2(2^{2^n}) - \log_2(n^n) \leq \log_2(F(n)) \leq 2^n \log_2(2)$$

$$2^n - n\log_2(n) \leq \log_2(F(n)) \leq 2^n$$

$$2^n \leq \log_2(F(n)) + n\log_2(n) \leq n\log_2(n) + 2^n$$

$$\log_2(F(n)) = 2^n + O(n\log_2(n))$$
$$= 2^n + O(n\log(n))$$

$$\boxed{\log_2(n) = \frac{\log(n)}{\log(2)}}$$
Log base change.