Monday Section (4/5/21)

- Section: Try to give a mini-lecture, then answer homework.
- I'll post recordings and notes on

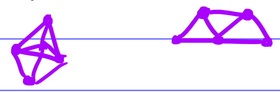
malachialexander.com/combinatorics/

Combinatorics is a broad area of research. At the heart, it pertains to discrete objects such as:

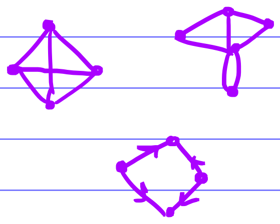
- countable sets

  $\{1,2,3\}$, $\mathbb{N}$, $\mathbb{Z}$

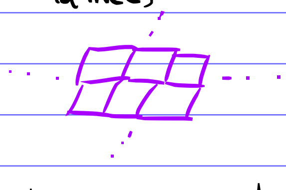
- simplicial complexes



- graphs



- discrete geometries
  - tilings
  - lattices



Classical combinatorics is focused on counting problems. However, it has grown to include applications in graph theory, probability theory, finite geometry, group theory, matroid theory and topology.

Because of the broadness of the subject, it may seem a bit disconnected. I tend to view combinatorics as a perspective on problems in mathematics and focus on what it solves more than it's actual purpose.

Asymptotic Growth (Big-O notation)

Let $f(x) = x^2 + 40$ domain on $\mathbb{N}$ ($\mathbb{R}$, or other sets)

$g(x) = x^3$

Which function grows faster?

$$\begin{array}{lcl} f(1) = 1^2+40 = 41 & > & g(1) = 1^3 = 1 \\ f(2) = 2^2+40 = 44 & > & g(2) = 2^3 = 8 \\ f(3) = 3^2+40 = 49 & > & g(3) = 3^3 = 27 \\ f(4) = 4^2+40 = 56 & < & g(3) = 4^3 = 64 \end{array}$$

and for all $n \geq 4$, $f(n) \leq g(n)$. However, we can't say $f \leq g$ because this means $f(n) \leq g(n)$ for all $n \in \mathbb{N}$. Need a bit more flexibility.

Definition: Let $f, g : \mathbb{N} \to \mathbb{N}$. We say $g(n) = O(f(n))$ if

- g asymptotically grows at most the same rate as f.
- $|g(n)| \leq c \cdot f(n)$ for some $c > 0$ and sufficiently large $n \in \mathbb{N}$.

Ex: Let $f(n) = n^2 + 10$ and $g(n) = 3n^2 + 25$

$$|3n^2+25| \leq 3(n^2+10) = 3n^2 + 30 \quad \text{for all } n \in \mathbb{N}.$$

Therefore, $3n^2+25 = O(n^2+10)$. Note, that replacing $c=3$ with any real number ≥ 3 will also work.

Ex: $\quad |3n^2+25| \leq 3.1(n^2+10) = 3.1n^2 + 31$

The point is, c is not necessarily unique.

It is more common (and more natural) to consider a polynomial like n^2 rather than n^2+10 in such a comparison as above

Ex: Let $f(n) = n^2$ and $g(n) = 3n^2+25$

$$|3n^2+25| \leq 5n^2 \quad \text{for} \quad n \geq 4$$
$$|3n^2+25| \leq 4n^2 \quad \text{for} \quad n \geq 5$$
$$|3n^2+25| \leq 3.5n^2 \quad \text{for} \quad n \geq 8$$
$$|3n^2+25| \leq 3.1n^2 \quad \text{for} \quad n \geq 16$$

In each case, we prove that $3n^2+25 = O(n^2)$. The point is that we don't necessarily need the inequality to hold for all $n \in \mathbb{N}$, but instead for all $n \geq m$ for some $m \in \mathbb{N}$.

Ex: Let $f(n) = n^3$ and $g(n) = 3n^2+25$.

$$|3n^2+25| \leq n^3 \quad \text{for } n \geq \underline{5}$$

$$28 = 3(1)^2+25 \geq 1^3 = 1$$
$$37 = 3(2)^2+25 \geq 2^3 = 8$$
$$52 = 3(3)^2+25 \geq 3^3 = 27$$
$$73 = 3(4)^2+25 \geq 4^3 = 64$$
$$100 = 3(5)^2+25 \leq 5^3 = 125$$

Thus, $3n^2+25 = O(n^3)$

Note: We obtained the following:

$$3n^2+25 = O(n^2) \quad \text{and} \quad 3n^2+25 = O(n^3)$$

How can it be equal to both? The equality "=" in this context is not the usual equals sign. The "=" means that $3n^2+25$ grows at most the same rate as n^2 and n^3.

$$3n^2+25 = O(n^k) \text{ for } k \geq 2.$$

Usage: Consider the expansion of $f(x) = x^5+x^3+2x+1$.

$$f(x) = x^5 + 7x^4 + 5x^2 + 7$$
$$= x^5 + 7x^4 + O(x^2)$$
$$= x^5 + O(x^4)$$
$$= O(x^5)$$

It allows for ambiguity.

Let $f, g: \mathbb{N} \to \mathbb{N}$

(1) $g(n) = O(f(n))$ $\leq$

- g asymptotically grows at most the same rate as f
- $|g(n)| \leq c f(n)$ for some $c > 0$ and sufficiently large n.

(2) $g(n) = \Omega(f(n))$ $\geq$

- g asymptotically grows at least the same rate as f
- $|g(n)| \geq c f(n)$ for some $c > 0$ and sufficiently large n.

(3) $g(n) = \Theta(f(n))$ $=$

- g asymptotically grows the same as f
- $g(n) = O(f(n))$ and $g(n) = \Omega(f(n))$

(4) $g(n) = o(f(n))$ "little oh" $<$

- g asymptotically grows slower than f
- $\lim_{n \to \infty} \frac{g(n)}{f(n)} = 0$

(5) $g(n) = \omega(f(n))$ "little omega" $>$

- g asymptotically grows faster than f
- $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$

Properties of Big-O. Let f, g be functions and c be a constant.

(1) $c \cdot f(n) = O(f(n))$

Multiplying by a constant does not change asymptotic growth.

(2) $O(f(n)) + O(g(n)) = O(f(n) + g(n)) = O(\max\{f(n), g(n)\})$

The asymptotic growth (for O) is dependent only on its fastest growing term.

Ex: $f(x) = 7x^5 + 2x^2 + 3$ $\qquad f(x) = O(x^5)$

(3) $O(f(n)) \cdot O(g(n)) = O(f(n) \cdot g(n))$

Ex: $f(x) = 7x^5 \qquad g(x) = (x+1)$

$f(x)g(x) = 7x^5(x+1) = 7x^6 + 7x^5$

$O(x^5) \cdot O(x) = O(x^6)$

5(a). $2^{2^3} = 256$ labelled families of subsets of a 3-set.

$X = \{a, b, c\}$

$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$

$\mathcal{P}'(X) = \{\emptyset, [1], [1], [1], [2], [2], [2], X\}$

Unlabeled, I can't distinguish between

$\{\{a\}\} \approx \{\{b\}\} \approx \{\{c\}\}$

$\{\{a\}, \{a,b\}\} \not\approx \{\{a\}, \{b,c\}\}$

when $\emptyset$ is in my family of subsets — 4 possibilities

X is in my family of subsets