Midterm 2 Version A Solutions

1. If $y = x^{13}e^x$, find $\frac{dy}{dx}$.

   Use Product Rule, $\frac{dy}{dx} = 13x^{12}e^x + e^x x^{13}$. Each term has a e^x and x^{12} in common, so we factor them out. So, $\frac{dy}{dx} = x^{12}e^x(13 + x)$.
   Answer: D.

2. Determine the points at which $f(x) = \frac{(x-1)(x+3)}{(x-1)(x-4)}$ is discontinuous.

   A rational function is discontinuous at holes and vertical asymptotes which occurs when the denominator is zero. Therefore, $(x - 1)(x - 4) = 0$ implies $x = 1$ or $x = 4$.
   Answer: A.

3. Find an equation for the line tangent to $y = \frac{36}{x^2+2}$ at $(1, 12)$.

   To find the slope of the tangent line, take the derivative and evaluate at 1. Use Chain Rule $y' = 36(-\frac{1}{(x^2+2)}) \cdot 2x = \frac{-72x}{(x^2+2)^2}$. At 1, $y'(1) = \frac{-72}{9} = -8$. To find y-intercept, plug into point-slope form.

   $$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 12 &= -8(x - 1) \\ y &= -8x + 20 \end{aligned}$$

   Answer: D.

4.
   $$\lim_{x\to 6^-} \frac{x-6}{|x-6|}$$

   When $x < 6$, $x - 6 < 0$, so the absolute value outputs the negative of the input.

   $$\lim_{x\to 6^-} \frac{x-6}{|x-6|} = \lim_{x\to 6^-} \frac{x-6}{-(x-6)} = \lim_{x\to 6^-} -1 = -1$$

   Answer: B.

5. If $y = \ln(x^5 - 3x - \pi)$, find $\frac{dy}{dx}$.

   Use Chain Rule, $y' = \frac{1}{x^5-3x-\pi} \cdot (5x^4 - 3) = \frac{5x^4-3}{x^5-3x-\pi}$.
   Answer: C.

6. Determine whether $f(x) = \frac{x^2-36}{x-6}$ is continuous at $c = 6$. If the function is not continuous, determine whether the discontinuity is removable or nonremovable.

   Directly substituting 6 into $f(x)$ gives us an undefined value (dividing by zero). Therefore, it is not continuous. Factor the top, $f(x) = \frac{x^2-36}{x-6} = \frac{(x-6)(x+6)}{x-6} = x + 6$. Since the denominator cancels with a term in numerator, this discontinuity is removable, meaning there is a hole at 6. If the hole was not there, then $f(6) = 12$ (we can see this by plugging in 6 into the reduced form).
   Answer: A.

7. Assuming that $x^6 = \cot(y)$ defines a differentiable function of x, find $\frac{dy}{dx}$ by implicit differentiation.

   Note that the derivative with respect to y of $\cot(y)$ is $-\csc^2(y)$. We can see this by rewriting $\cot(y) = \frac{cos(y)}{\sin(y)}$ and use the Quotient Rule. We take the derivative with respect to x on both sides.

$$\frac{d}{dx}[x^6] = \frac{d}{dx}[\cot(y)]$$
$$6x^5 = -\csc^2(y)\frac{dy}{dx}$$
$$\frac{6x^5}{-\csc^2(y)} = \frac{dy}{dx}$$

   Answer: B.

8. If $y = e^{3x^2-3x}$, find $\frac{dy}{dx}$.

   Use Chain Rule, $e^{3x^2-3x} \cdot (6x - 3)$
   Answer: A.

9. The $\lim_{x\to 9} \frac{x^2-81}{x-9}$ is a derivative, but of what function and at what point?

   Note that $\frac{x^2-81}{x-9} = \frac{x^2-(9)^2}{x-9}$. We are taking the inputs and squaring them and evaluating the limit at 9. Therefore, $f(x) = x^2$ at $x = 9$.
   Answer: A.

10. If $y = \frac{3x+3}{9x-2}$, find $\frac{dy}{dx}$.

    Use the Quotient Rule, $y' = \frac{(9x-2)3-(3x+3)9}{(9x-2)^2} = \frac{(27x-6)-(27x+27)}{(9x-2)^2} = \frac{-33}{(9x-2)^2}$.
    Answer: D.

11. Let $\lim_{x\to 10} f(x) = 81$ and $\lim_{x\to 10} g(x) = 4$. Find $\lim_{x\to 10} \sqrt[4]{f(x)}[g(x)+1]$.
Use limit rules,

$$\lim_{x\to 10} \sqrt[4]{f(x)}[g(x)+1] = \sqrt[4]{\lim_{x\to 10} f(x)}[\lim_{x\to 10} g(x) + \lim_{x\to 10} 1] = \sqrt[4]{81}[4+1] = 15$$

Answer: B.

12.

$$\lim_{t\to 9^+} \frac{t^2}{81-t^2}$$

If we approach from the right $9^2 - t^2 < 0$ since $t > 9$. As t approaches 9, the denominator becomes smaller and smaller, making the value of the function become large. So the limit is $-\infty$.
Answer: C.

13.

$$\lim_{x\to\infty} \frac{6x^3+1}{10x^2-\pi x^3}$$

Multiply top and bottom by $\frac{1}{x^3}$, then

$$\lim_{x\to\infty} \frac{6x^3+1}{10x^2-\pi x^3} = \lim_{x\to\infty} \frac{6+\frac{1}{x^3}}{\frac{10}{x}-\pi} = \frac{6+\frac{1}{\infty}}{\frac{10}{\infty}-\pi} = \frac{6}{-\pi}$$

Answer: D.

14.

$$\lim_{x\to 4} \frac{x^2-16}{x^2-5x+4}$$

Factor then evaluate the limit.

$$\lim_{x\to 4} \frac{x^2-16}{x^2-5x+4} = \lim_{x\to 4} \frac{(x+4)(x-4)}{(x-4)(x-1)} = \lim_{x\to 4} \frac{x+4}{x-1} = \frac{8}{3}$$

Answer: A.

15. Let $\lim_{x\to -8} f(x) = 7$ and $\lim_{x\to -8} g(x) = 9$. Find $\lim_{x\to -8} \frac{f(x)}{g(x)}$.
Use limit rules,

$$\lim_{x\to -8} \frac{f(x)}{g(x)} = \frac{\lim_{x\to -8} f(x)}{\lim_{x\to -8} g(x)} = \frac{7}{9}$$

Answer: D.

16. If $y = x^{9-e}$, find $\frac{dy}{dx}$.
Use Chain Rule. $y' = (9-e)x^{9-e-1} = (9-e)x^{8-e}$.
Answer: A.

17. Find an equation for the line tangent to $y = x + \frac{1}{x}$, when $x = 2$.

    To find the slope of the tangent line, take the derivative and evaluate at 2, $y' = 1 - \frac{1}{x^2}$. At 2, $y'(2) = 1 - \frac{1}{4} = \frac{3}{4}$. We need the corresponding y-value, so we plug into the original equation. $y(2) = 2 + \frac{1}{2} = \frac{5}{2}$. Now plug into the point-slope form and simplify.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - \frac{5}{2} &= \frac{3}{4}(x - 2) \\
y &= \frac{3}{4}x - \frac{3}{2} + \frac{5}{2} \\
y &= \frac{3}{4}x + 1
\end{aligned}
$$

    Answer: B.

18. Given that $f(x) = \frac{4}{x}$, find $\lim_{x \to 6} \frac{f(x) - f(6)}{x - 6}$.
    This is the definition of a derivative, therefore, take the derivative of $f(x)$ and evaluate at 6, $f'(x) = -\frac{4}{x^2}$. At 6, $f'(6) = -\frac{4}{36} = -\frac{1}{9}$.
    Answer: B.

19. State whether $f(t)$ is continuous at the point $t = 5$.

$$
f(t) \begin{cases} 8t - 8 & \text{if } t \neq 5 \\ -12 & \text{if } t = 5 \end{cases}
$$

    If I take the limit as t approaches 5, we get that $f(t)$ approaches $32 \neq -12$. So $f(t)$ is not continuous.
    Answer: A.

20. If $y = x^9 \ln(x)$, find $\frac{dy}{dx}$. Use the Product Rule, $y' = 9x^8 \ln(x) + \frac{1}{x} \cdot x^9 = 9x^8 \ln(x) + x^8$.
    Answer: B.