Related Rates

1. A small water droplet falls into a pond, causing ripples of the form of a circle. The radius r of the outermost ripple is increasing at a constant rate of 3 feet per second. When the radius is at 5 feet, what rate is the total area A of disturbed water changing? Hint: Area of a circle is $A = \pi r^2$.

   Solution 1. We relate the radius and area of a circle via the equation $A = \pi r^2$. The rate of change of the radius r is $\frac{dr}{dt} = 3$. We want to find $\frac{dA}{dt}$ when the radius is 5 feet. Take the derivative of the equation with respect to t in seconds:

$$\begin{aligned} A &= \pi r^2 \\ \frac{dA}{dt} &= 2\pi r \frac{dr}{dt} \end{aligned}$$

   Therefore, by substituting the appropriate values:

$$\begin{aligned} \frac{dA}{dt} &= 2\pi(5)(3) \\ &= 30\pi \end{aligned}$$

2. We are inflating a spherical balloon at a rate of 2 cubic inches per second. Find the rate of change of the radius when the radius is 8 inches. Hint: Volume of a sphere is $V = \frac{4}{3}\pi r^3$.

   Solution 2. We relate the radius of the sphere to the volume via the equation $V = \frac{4}{3}\pi r^3$. The rate of change of the volume V is $\frac{dV}{dt} = 2$. We want to find $\frac{dr}{dt}$ when the radius is 8 inches. Take the derivative of the equation with respect to t in seconds:

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt} \end{aligned}$$

   Therefore, by substituting the appropriate values:

$$\begin{aligned} 2 &= 4\pi(8)^2 \frac{dr}{dt} \\ \frac{2}{4\pi(8)^2} &= \frac{dr}{dt} \\ \frac{1}{128\pi} &= \frac{dr}{dt} \end{aligned}$$

3. A drone is on a straight flight path that takes it directly over a spectator while maintaining an altitude of 1 kilometer. If the distance l is decreasing at a rate of 20 kilometers per hour when the drone is 2 kilometers away, what is the speed of the drone?

Solution 3. We want to relate the distance from the drone to the spectator. Since the drone will be flying right above the spectator, and we are at a fixed altitude, we can relate the distance to a triangle. We can think of the distance l as the hypotenuse, solve for the altitude (since it will be fixed) and introduce a new variable h for the horizontal distance from the spectator. We use the equation $h^2 + a^2 = l^2$, where a is the altitude to relate the unknowns. When the plane is 2 kilometers away,

$$\begin{aligned} h^2 + 1^2 &= 2^2 \\ h^2 &= 3 \\ h &= \sqrt{3} \end{aligned}$$

We take the derivative to get the corresponding rates of change, note that the altitude is fixed:

$$\begin{aligned} h^2 + 1 &= s^2 \\ 2h\frac{dh}{dt} &= 2l\frac{dl}{dt} \\ 2(\sqrt{3})(-20) &= 2(2)\frac{dl}{dt} \\ -10\sqrt{3} &= \frac{dl}{dt} \end{aligned}$$