Quiz 5

1. Find $f'(x)$ when $f(x) = \sqrt{x}(x^2 + \ln(3))$.
2. Find $g'(x)$ when $g(x) = \frac{\sin(x)}{x^2}$.
3. Find the tangent and normal line at $x = 0$ for the function $h(x) = e^x + x$.
4. Find $(fg)'(-\pi)$ when $f(x) = 2x^3$ and $g(x) = \tan(x)$.

Solutions to Quiz 5.

1. There are two ways to approach this problem, the first approach is to distribute $\sqrt{x}$ before computing the derivative.

$$\begin{aligned} f(x) &= \sqrt{x}(x^2+\ln(3)) \\ &= \sqrt{x}x^2 + \sqrt{x}\ln(3) \\ &= x^{\frac{5}{2}} + \ln(3)x^{\frac{1}{2}} \\ f'(x) &= \frac{5}{2}x^{\frac{3}{2}} + \ln(3)\frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{5x^{\frac{3}{2}}}{2} + \frac{\ln(3)}{2\sqrt{x}} \\ &= \frac{5x^2+\ln(3)}{2\sqrt{x}} \end{aligned}$$

Alternatively, we can use the product rule.

$$\begin{aligned} f(x) &= \sqrt{x}(x^2+\ln(3)) \\ &= x^{\frac{1}{2}}(x^2+\ln(3)) \\ f'(x) &= \frac{1}{2}x^{-\frac{1}{2}}(x^2+\ln(3)) + x^{\frac{1}{2}}(2x) \\ &= \frac{x^2+\ln(3)}{2\sqrt{x}} + 2x^{\frac{3}{2}} \\ &= \frac{x^2+\ln(3)+4x^2}{2\sqrt{x}} \\ &= \frac{5x^2+\ln(3)}{2\sqrt{x}} \end{aligned}$$

2. There are two approaches to this problem, the first is to use the quotient rule

$$\begin{aligned} f(x) &= \frac{\sin(x)}{x^2} \\ f'(x) &= \frac{x^2\cos(x) - \sin(x)(2x)}{(x^2)^2} \\ &= \frac{x^2\cos(x) - 2x\sin(x)}{x^4} \\ &= \frac{x\cos(x) - 2\sin(x)}{x^3} \end{aligned}$$

The second approach is to use exponent rules and then apply the product rule:

$$\begin{aligned}
f(x) &= \frac{\sin(x)}{x^2} \\
&= x^{-2}\sin(x) \\
f'(x) &= -2x^{-3}\sin(x) + x^{-2}\cos(x) \\
&= \frac{-2\sin(x)}{x^3} + \frac{\cos(x)}{x^2} \\
&= \frac{-2\sin(x) + x\cos(x)}{x^3} \\
&= \frac{x\cos(x) - 2\sin(x)}{x^3}
\end{aligned}$$

3. To find the tangent and normal line at $x = 0$, we compute the derivative of h.

$$h'(x) = e^x + 1$$

To find the slope of the tangent line at $x = 0$, we compute $h'(0) = 2$. To find the y-intercept, we need a coordinate. Therefore, we compute $h(0) = 1$ to obtain $(0, 1)$. Therefore,

$$\begin{aligned}
y &= 2x + b \\
1 &= 2(0) + b \\
1 &= b
\end{aligned}$$

Therefore, the tangent line is given by $y = 2x + 1$. The normal line has slope $-\frac{1}{2}$ (the negative reciprocal of 2). Therefore,

$$\begin{aligned}
y &= -\frac{1}{2}x + b \\
1 &= -\frac{1}{2}(0) + b \\
1 &= b
\end{aligned}$$

Therefore, the normal line is given by $y = -\frac{1}{2}x + 1$.

4. Note that

$$(fg)'(-\pi) = f(-\pi)g'(-\pi) + f'(-\pi)g(-\pi)$$

We do the following computations:

$$\begin{aligned}
f(x) &= 2x^3 & g(x) &= \tan(x) \\
f'(x) &= 6x^2 & g'(x) &= \sec^2(x)
\end{aligned}$$

Therefore,

$$f(-\pi) = -2\pi^3 \qquad\qquad g(x) = \tan(\pi) = 0$$
$$f'(-\pi) = 6\pi^2 \qquad\qquad g'(x) = \sec^2(-\pi) = 1$$

Therefore,

$$(fg)'(-\pi) = f(-\pi)g'(-\pi) + f'(-\pi)g(-\pi) = (-2\pi^3)(1) + (6\pi^2)(0)$$

Therefore, $(fg)'(-\pi) = -2\pi^3$. Alternatively, we can compute the product first, then evaluate.