Topology of Positive Zero Sets of Bivariate Pentanomials

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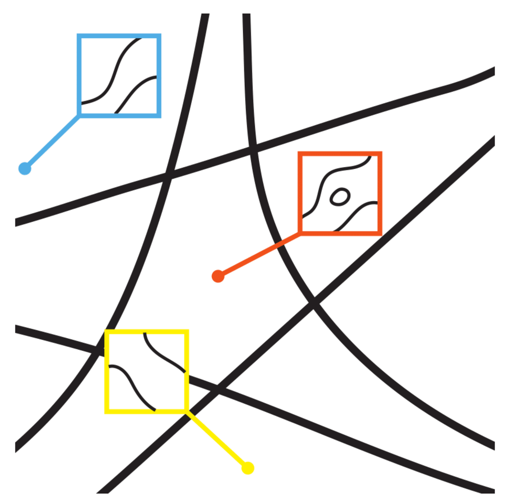
Motivation

The $\mathcal{A}$-discriminant describes the set of polynomials p, with a fixed support $\mathcal{A}$, having a degenerate zero set. For certain $\mathcal{A}$, we get a curve (consisting of projections of coefficient vectors) and its complement consists of *discriminant chambers*. The topology of the zero set of p changes only when the coefficient vector enters a new chamber. We attempt to automate topology computation this way in the first non-trivial case of n-variate (n+3)-nomials: bivariate pentanomials.



Bivariate Pentanomials

Definition: A polynomial $p(x)$ of two variables and five terms, where $c_i \in \mathbb{R}$ and $a_i \in \mathbb{R}^2$.

$$p(x) = c_1x^{a_1} + c_2x^{a_2} + c_3x^{a_3} + c_4x^{a_4} + c_5x^{a_5}$$

Example:

$$p_1(x) = 10x_1^2x_2^2 - 8x_1x_2^8 - 11x_1^5x_2^8 + 8x_1^8x_2^4 + 3x_1^4x_2$$

Computing B-Matrix

Let $\mathcal{A} = [a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5]$ and $\widehat{\mathcal{A}} = \begin{bmatrix} 1 \\ \mathcal{A} \end{bmatrix}$.

We define B such that $\widehat{\mathcal{A}}B = \mathbf{O}$, in other words, B is a right null space of $\widehat{\mathcal{A}}$.

Example:

From $p_1(x)$, we have $\mathcal{A}_1 = \begin{bmatrix} 2 & 1 & 5 & 8 & 4 \\ 2 & 8 & 8 & 4 & 1 \end{bmatrix}$, so,

$$\widehat{\mathcal{A}}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 5 & 8 & 4 \\ 2 & 8 & 8 & 4 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -8 & 15 & -19 & 12 & 0 \\ -28 & 15 & -11 & 0 & 24 \end{bmatrix}$$

Horn-Kapranov Uniformization

Using the B-matrix, we are able to parametrize the discriminant curve using the Horn-Kapranov Uniformization:

$$\varphi(\lambda) := (\mathrm{Log}|\lambda B^T|)B,$$

where $\lambda = (\lambda_1, \lambda_2)$. Using this equation, we set $\lambda_1 = \sin(\theta)$ and $\lambda_2 = \cos(\theta)$ such that $0 < \theta \leq \pi$ and parameterize our $\mathcal{A}$-discriminant curve along the projective semi-circle.

Definitions

Newton Polytope ($\mathrm{Newt}(p)$)

$\mathrm{Newt}(p) := \mathrm{Conv}(a_1, \ldots, a_5)$

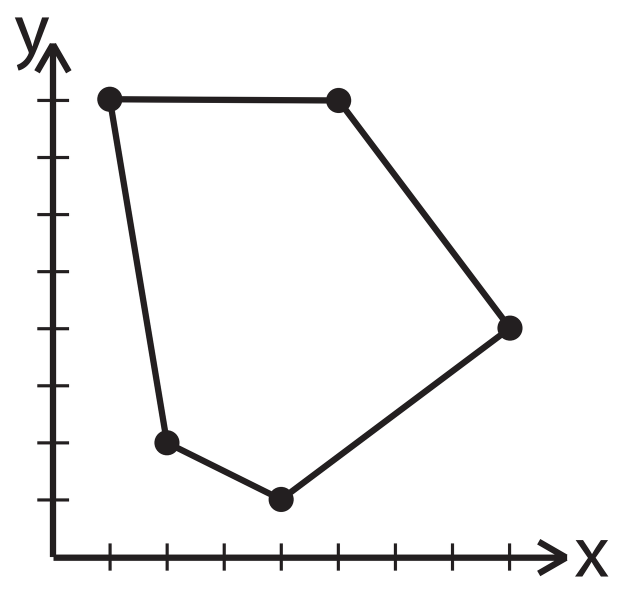


Figure 1: $\mathrm{Newt}(p_1)$

Archimedean Newton Polytope($\mathrm{ArchNewt}(p)$)

$\mathrm{ArchNewt}(p) := \mathrm{Conv}(\{(a_i, -\mathrm{Log}|c_i|) \mid i \in \{1, \ldots, 5\}\})$

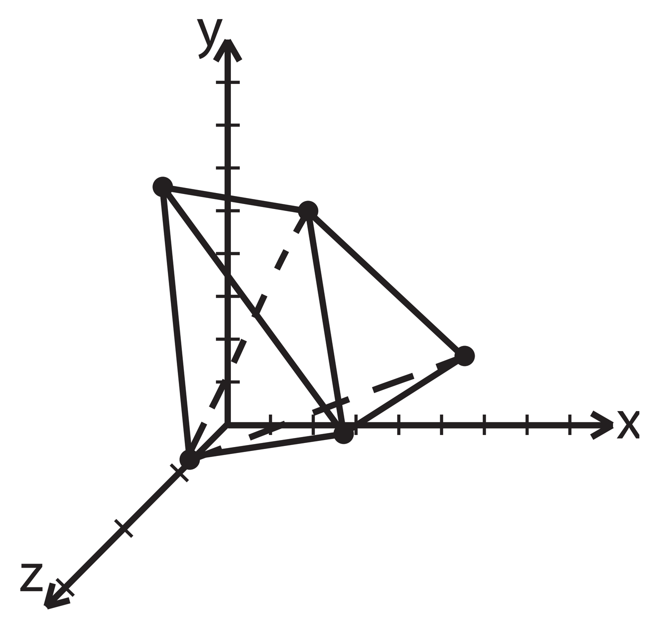


Figure 2: $\mathrm{ArchNewt}(p_1)$

Positive Tropical Variety ($\mathrm{Trop}_+(p)$)

$\mathrm{Trop}_+(p) := \{v \in \mathbb{R}^2 \mid (v, -1)$ outer normal to a face of $\mathrm{ArchNewt}(p)$ with $c_ic_j < 0\}$

Note: We can think of computing $\mathrm{Trop}_+(p)$ as using $\mathrm{Log}|c_i|$ as a lifting function. The result of this lift is $\mathrm{ArchNewt}(p)$. We then triangulate $\mathrm{ArchNewt}(p)$, focus on the outer normals of the lower hull and construct rays and edges given the condition $c_ic_j < 0$ is satisfied.

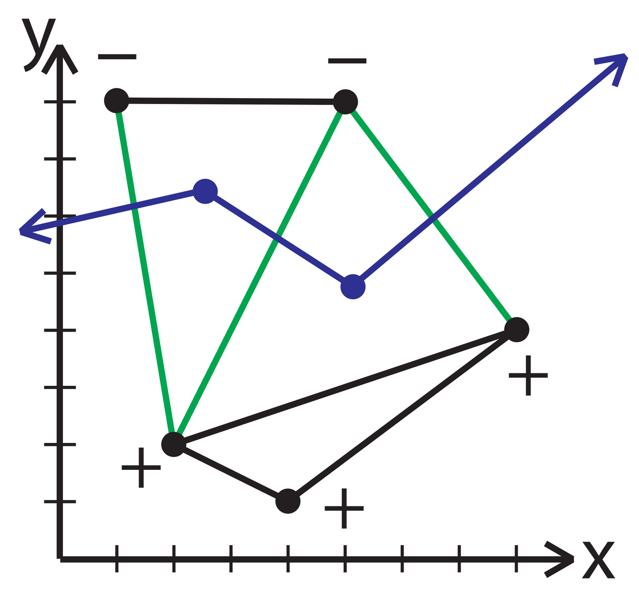


Figure 3: Approximation of $\mathrm{Trop}_+(p_1)$

Theorem

If $p \in \mathbb{R}[x_1, x_2]$ has coefficient vector c and $\mathrm{Log}|c|B$ the point lying in an outer chamber of the $\mathcal{A}$-discriminant, then $\mathrm{Trop}_+(p)$ and $Z_+(p)$ are isotopic.

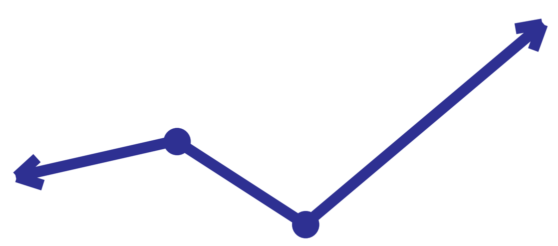


Figure 4: Topology of Positive Tropical Variety

Figure 5: Topology of Positive Zero Set

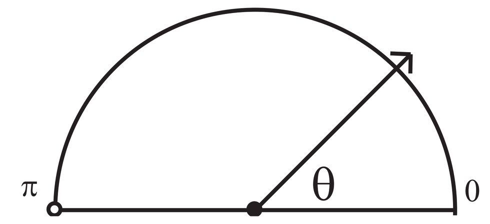
Algorithm

1. Compute B-Matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix} B = \mathbf{O}$$

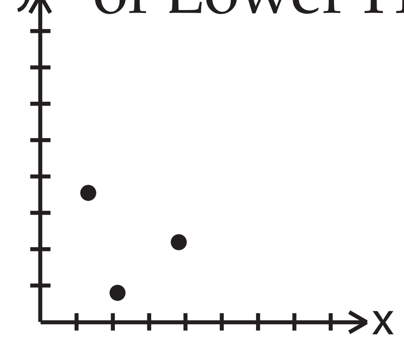
2. Use Horn-Kapranov Uniformization

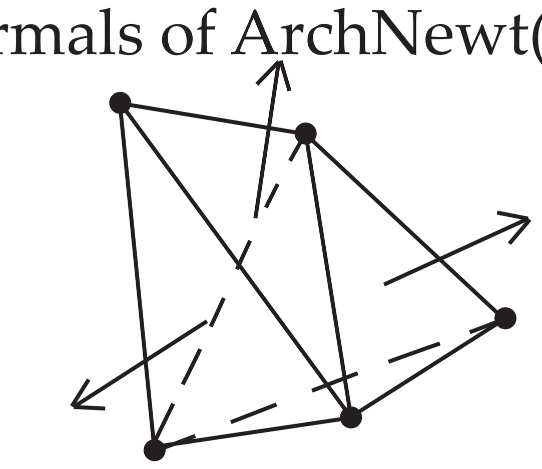
$\varphi(\lambda) = \mathrm{Log}(\lambda B^T)B$



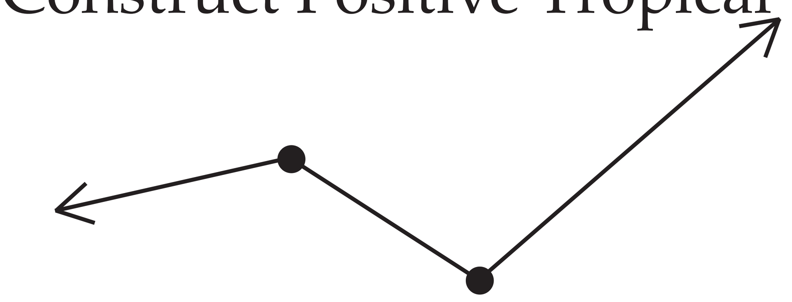
3. Graph Discriminant Curve

4. Find Outer Normals of $\mathrm{ArchNewt}(p)$ of Lower Hull





5. Construct Positive Tropical Variety



Future Work

1. Compute topology of inner chambers of the $\mathcal{A}$-discriminant.
2. Extend the algorithm to all n-variate (n+3)-nomials

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