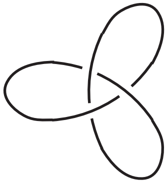
Hexagonal Mosaic Knots

Malachi Alexander, Selina Foster, and Gianni Krakoff
Mentored by Dr. Jennifer McLoud-Mann

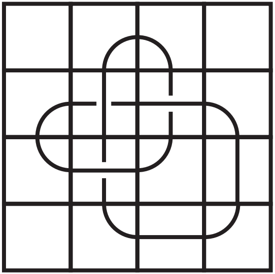
University of Washington Bothell
Research Experience for
Undergraduates

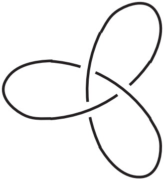
August 28, 2017

What is a Mosaic Knot?

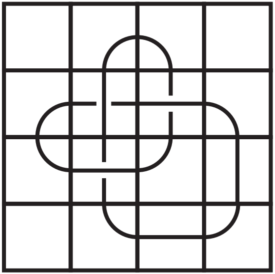


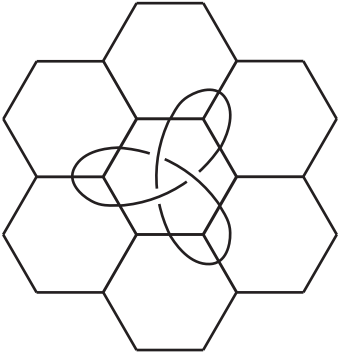
What is a Mosaic Knot?





What is a Mosaic Knot?



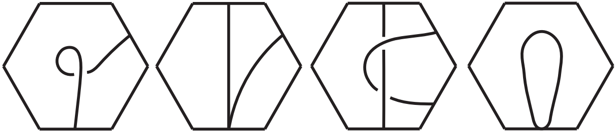


Defining the Hextile

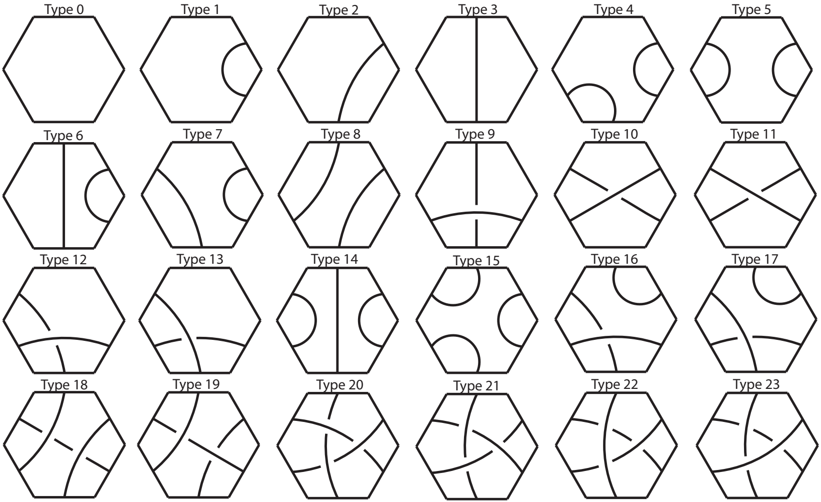
A hextile must obey the following axioms:

- A curve must terminate at the midpoint of an edge and a curve cannot cross itself.
- Two curves cannot cross more than once and cannot share an edge.

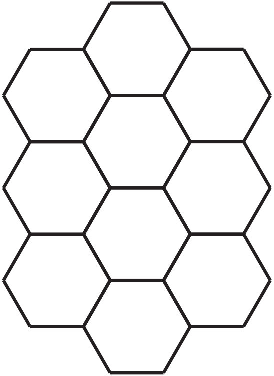
Examples of Violations



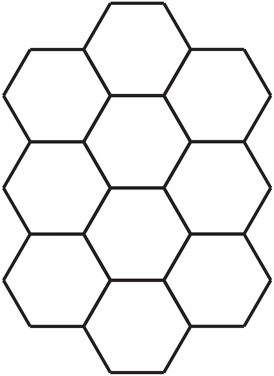
Types of Hextiles



Arrangements and Diagrams

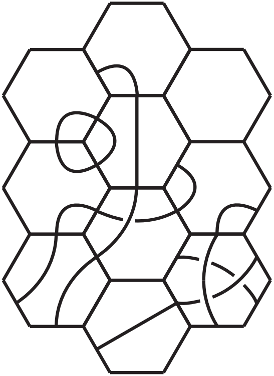


Arrangement of ten hextiles.

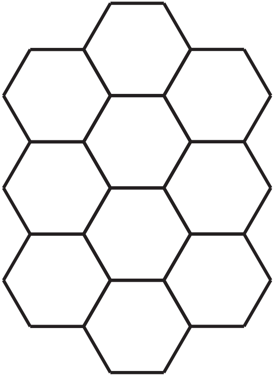


Arrangement of ten hextiles.

Arrangements and Diagrams

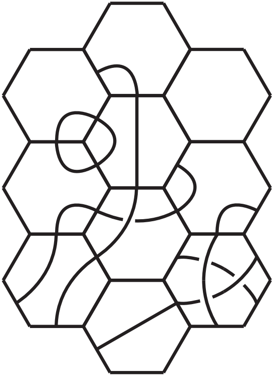


Not suitably connected.

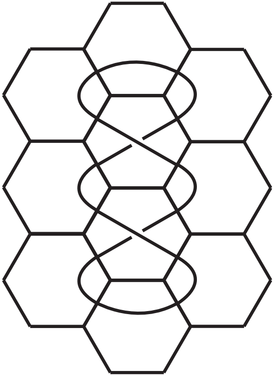


Arrangement of ten hextiles.

Arrangements and Diagrams

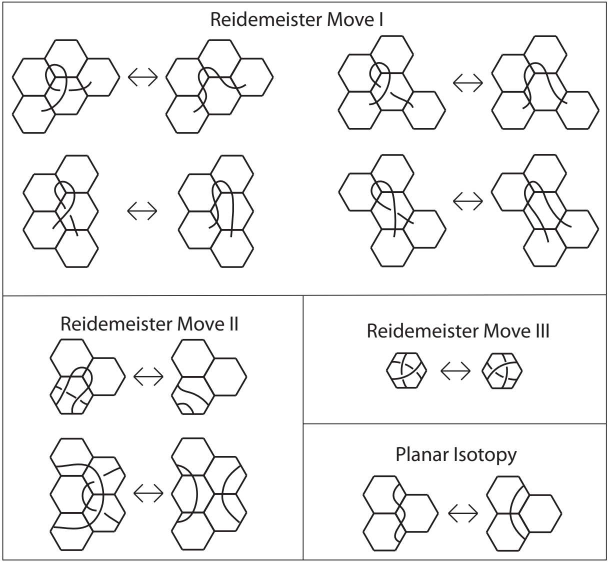


Not suitably connected.



Suitably connected.

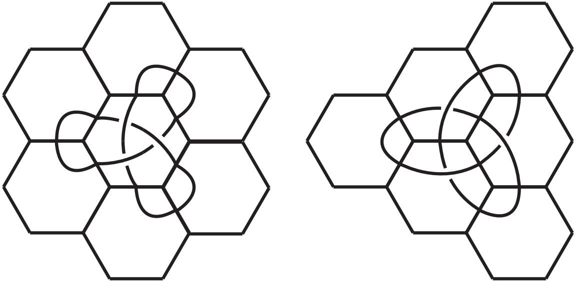
Reidemeister Moves & Planar Isotopy



The Hextile Number

Definition

The *hextile number* of a link L is the least number of hextiles needed to represent L, denoted $h(L)$.



More crossings per tile does not imply hextile number.

Hextile Number as a Knot Invariant

Definition

The *hextile number* of a link L is the least number of hextiles needed to represent L, denoted $h(L)$.

Theorem

The hextile number is knot invariant.

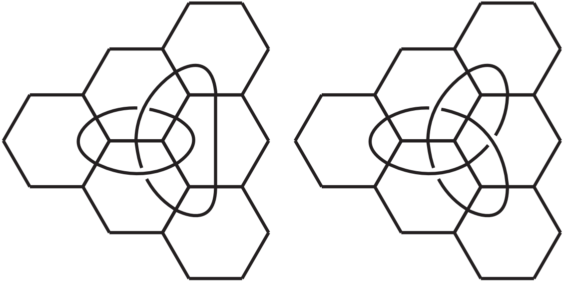
Proof.

Similar to the crossing number, given two knots if the hextile numbers are different then the knots must be different, and if the hextile numbers are the same then we can't conclude the knots are different. □

Theorems About Hextile Number

Theorem

For a non-trivial link L, $h(L) \geqslant 6$.



$h(2_1^2) = 6$ and $h(3_1) = 6$.

Proof Concept: Pincer Movement

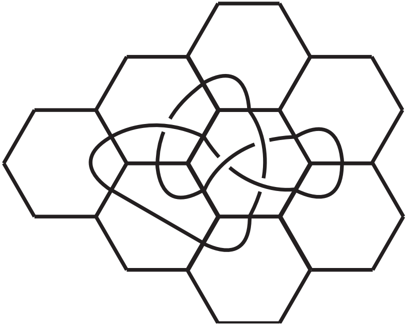
Construction

- To show that we actually have in our hands the hextile number for some knot, our main technique has been squeezing the upper and lower bounds.
- We want to show that $h(3_1) = 6$; it is sufficient to show that $h(3_1) > 5$, and that $7 > h(3_1)$. We already have it on 6.
- Computations become exponentially harder as the number of hextiles increases.

Theorems About Hextile Number and Crossing Number

Theorem

For a link L, if $c(L) \geqslant 4$, then $h(L) \geqslant 8$.

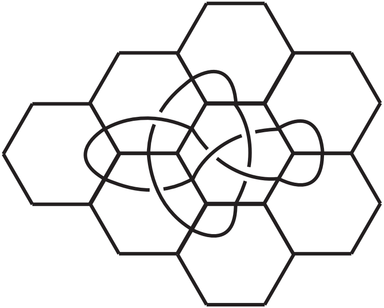


$h(4_1) = 8.$

Theorems About Hextile Number and Crossing Number

Theorem

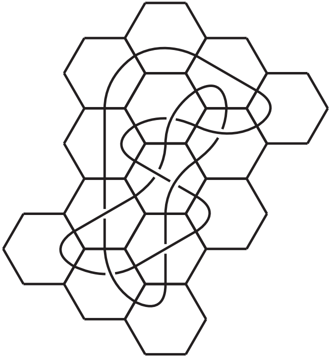
For a knot K, if $c(K) \geqslant 5$, then $h(K) \geqslant 9$.



The Whitehead link on eight hextiles.

Known and Unknown Hextile Numbers

L	$h(L)$
0_1	3
0_1^2	5
2_1^2	6
3_1	6
4_1	8
4_1^2	8
5_1^2	8
5_1	9
5_2	9
$3_1 \# 3_1$	9



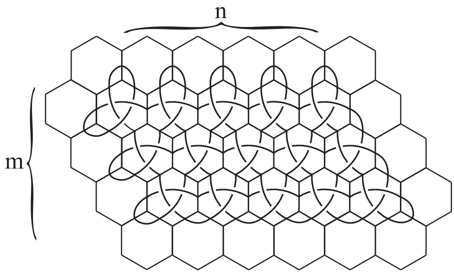
Is this $D(7_3)$ reducible?

L	$h(L)$
6_1	9
6_2	9
6_3	9
7_1	12?
7_2	13?
7_3	14?
7_4	11?
7_5	11?
7_6	10?
7_7	9

Saturation: Terminology & Construction

Definition

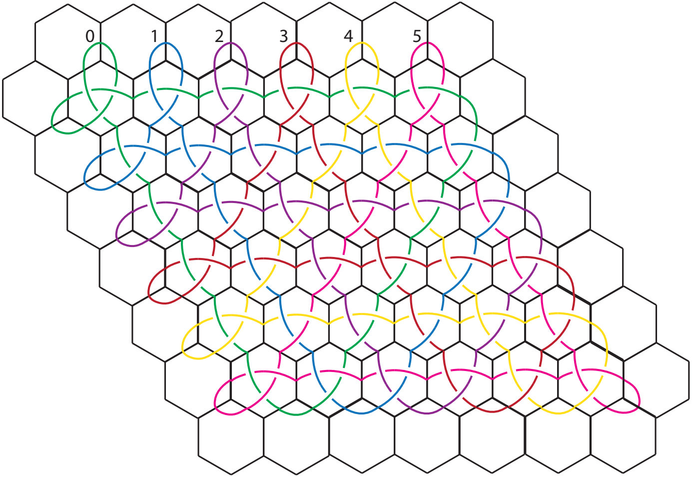
A diagram is called *saturated* if every interior hextile is a three-crossing hextile.



Construction

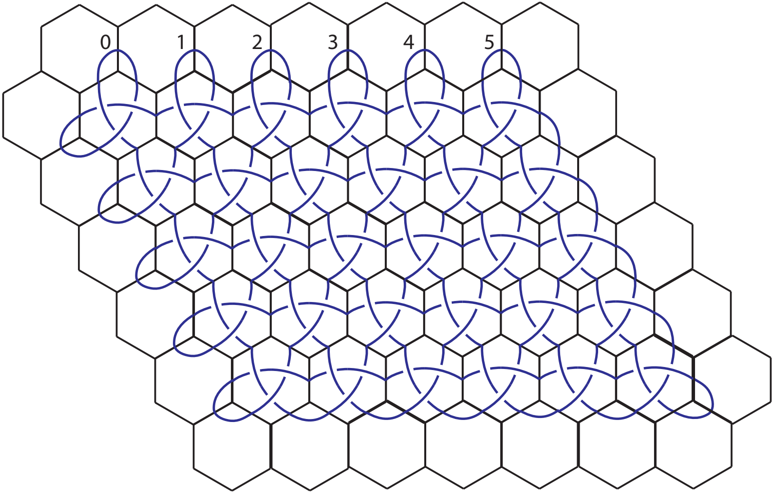
Place T_{20} or T_{21} in an $m \times n$ parallelogram, then suitably connect without nugatory crossings.

Examples



6 × 6 with six distinct components.

Examples

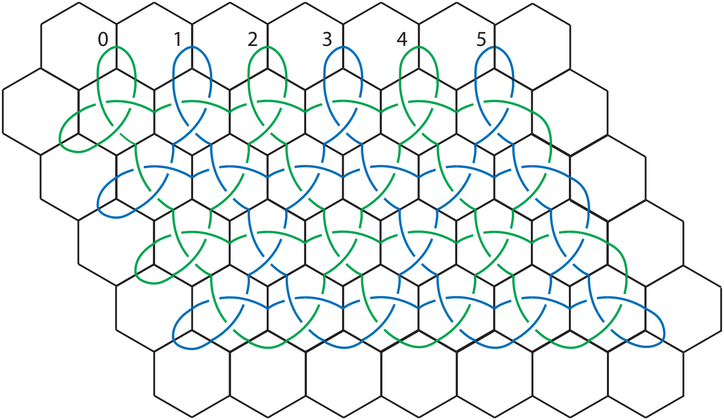


5×6 with one component.

Algebraic Structure

Observation

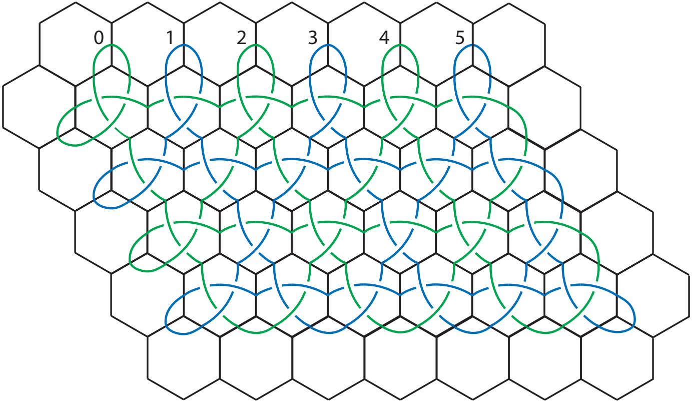
Each component is represented by the distinct cosets of $\langle \gcd(m, n) \rangle$ in $\mathbb{Z}_n$.



Algebraic Structure

Theorem

$D(L)$ is a reduced and alternating link with $\gcd(m, n)$ components. Therefore L is a knot if and only if m and n are relatively prime.



Thank You!

We would like to thank:

- Our mentor, Dr. Jennifer McLoud-Mann
- All the mentors of The University of Washington Bothell REU
- The National Science Foundation Grant DMS1460699

References

[1] ADAMS, C.
The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.
Berlin; New York: American Mathematical Society, 2004.

[2] GALLIAN, J. A.
Contemporary Abstract Algebra.
Houghton Mifflin, Boston, New York, 2006.

[3] HOWARDS, H., AND KOBIN, A.
Crossing number bound in knot mosaics.
arXiv:1405.7683 (2014).

[4] LOMONACO, S. J., AND KAUFFMAN, L. H.
Quantum knots and mosaics.
Quantum Information Processing 7, 2-3 (2008), 85–115.

[5] LUDWIG, L. D., EVANS, E. L., AND PAAT, J. S.
An infinite family of knots whose mosaic number is realized in non-reduced projections.
Journal of Knot Theory and Its Ramifications 22, 07 (2013), 1350036.

Further Work

Open Questions

- Find hextile numbers for the remaining seven-crossing knots and higher crossing knots.
- Is there a bound for hextile number in terms of crossing number, or other known knot invariants?
- Are all saturated links realizing their hextile number?